CALCULUS AB
SECTION II, Part B
Time—60 minutes
Number of problems—4

No calculator is allowed for these problems.

t (minutes)	0	12	20	24	40
$v(t)$ (meters per minute)	0	200	240	−220	150

3. Johanna jogs along a straight path. For $0 \le t \le 40$, Johanna's velocity is given by a differentiable function v. Selected values of $v(t)$, where t is measured in minutes and $v(t)$ is measured in meters per minute, are given in the table above.

(a) Use the data in the table to estimate the value of $v'(16)$.

(b) Using correct units, explain the meaning of the definite integral $\int_0^{40} |v(t)|\, dt$ in the context of the problem. Approximate the value of $\int_0^{40} |v(t)|\, dt$ using a right Riemann sum with the four subintervals indicated in the table.

(c) Bob is riding his bicycle along the same path. For $0 \le t \le 10$, Bob's velocity is modeled by $B(t) = t^3 - 6t^2 + 300$, where t is measured in minutes and $B(t)$ is measured in meters per minute. Find Bob's acceleration at time $t = 5$.

(d) Based on the model B from part (c), find Bob's average velocity during the interval $0 \le t \le 10$.

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2015 RELEASED FREE RESPONSE SOLUTIONS – MR. CALCULUS

2015 AB/BC #3
(no calculator)

t (minutes)	0	12	20	24	40
$v(t)$ (meters per minute)	0	200	240	−220	150

(a)

$$v'(16) \approx \frac{v(20)-v(12)}{20-12} = \boxed{\frac{240-200}{20-12}} \ \frac{m}{min^2} = 5 \ \frac{m}{min^2}$$

(b)

$\int_0^{40} |v(t)|\,dt$ is the total distance that Johanna jogs, in meters, from time $t = 0$ min to $t = 40$ min

$$\int_0^{40} |v(t)|\,dt \approx (12-0)|v(12)| + (20-12)|v(20)| + (24-20)|v(24)| + (40-24)|v(40)|$$

$$= \boxed{(12-0)200 + (20-12)240 + (24-20)220 + (40-24)150} \text{ meters} = 7600 \text{ m}$$

(c)

$$a(t) = B'(t) = 3t^2 - 12t \quad a(5) = B'(5) = \boxed{3(5)^2 - 12(5)} \ \frac{m}{min^2} = 15 \ \frac{m}{min^2}$$

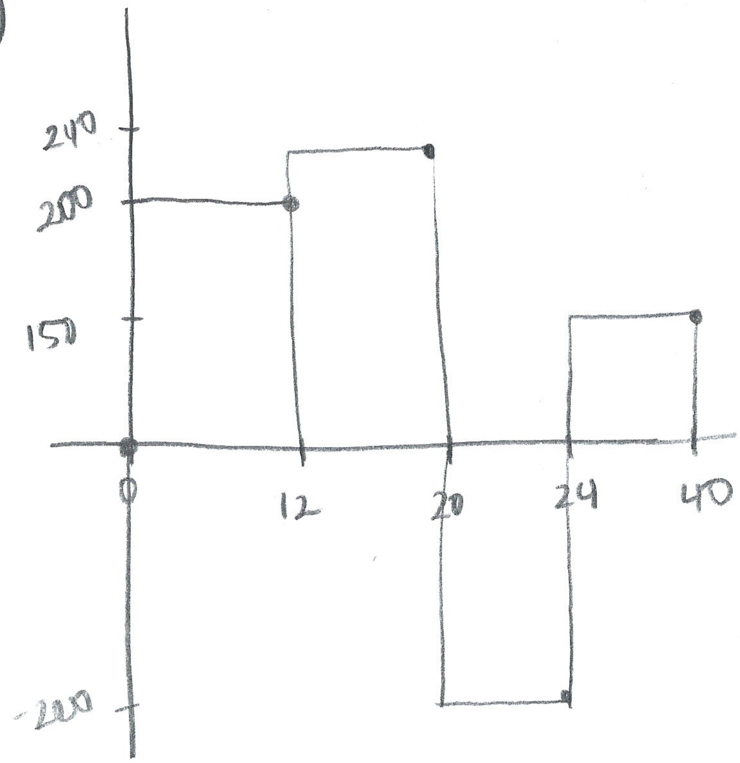
(d)

$$\boxed{\text{velocity}_{avg} = \frac{1}{10-0}\int_0^{10} B(t)\,dt = \frac{1}{10}\left[\frac{1}{4}t^4 - 2t^3 + 300t\right]_0^{10} = \boxed{\frac{1}{10}\left[\frac{1}{4}(10)^4 - 2(10)^3 + 300(10)\right]} = 35}$$

a) $V'(16) = \dfrac{y_2 - y_1}{x_2 - x_1}$ $(12, 200)$ and $(20, 240)$

$V'(16) = \dfrac{240-200}{20-12} = \dfrac{40}{8} = \boxed{5}$ m/min²

b)



$12(200) + 8(240)$

$+ 4(220) + 16(150) =$

$= 2400 + 1920 + 880 + 2400 =$

$\boxed{7600\text{m}}$

c) $B(t) = V(t) = t^3 - 6t^2 + 300$

$a(t) = 3t^2 - 12t$

$a(5) = 3(5^2) - 12(5) = 75 - 60 = \boxed{15\text{m/min}^2}$

d) average velocity = average value

$\frac{1}{10-0}\int_0^{10} B(t) = \frac{1}{10}\int_0^{10} t^3 - 6t^2 + 300$

$= \frac{1}{10}\left[\frac{t^4}{4} - \frac{6t^3}{3} + 300t\right]_1^{10} = \frac{1}{10}\left(\frac{10^4}{4} - 2(10)^3 + 300(10)\right)$

$= \boxed{35}$

TABULAR DATA AND REIMANN SUM

2013 AP® CALCULUS BC FREE-RESPONSE QUESTIONS

t (minutes)	0	1	2	3	4	5	6
$C(t)$ (ounces)	0	5.3	8.8	11.2	12.8	13.8	14.5

3. Hot water is dripping through a coffeemaker, filling a large cup with coffee. The amount of coffee in the cup at time t, $0 \le t \le 6$, is given by a differentiable function C, where t is measured in minutes. Selected values of $C(t)$, measured in ounces, are given in the table above.

(a) Use the data in the table to approximate $C'(3.5)$. Show the computations that lead to your answer, and indicate units of measure.

(b) Is there a time t, $2 \le t \le 4$, at which $C'(t) = 2$? Justify your answer.

(c) Use a midpoint sum with three subintervals of equal length indicated by the data in the table to approximate the value of $\frac{1}{6}\int_0^6 C(t)\,dt$. Using correct units, explain the meaning of $\frac{1}{6}\int_0^6 C(t)\,dt$ in the context of the problem.

(d) The amount of coffee in the cup, in ounces, is modeled by $B(t) = 16 - 16e^{-0.4t}$. Using this model, find the rate at which the amount of coffee in the cup is changing when $t = 5$.

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a) $C'(3.5) \rightarrow$ slope at 3.5

use table

t	3	4
$C(t)$	11.2	12.8

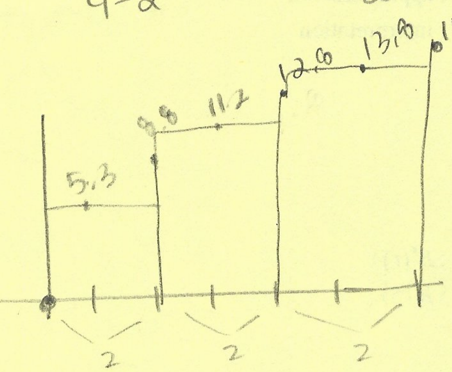
$\frac{y_2 - y_1}{x_2 - x_1} = \frac{12.8 - 11.2}{4-3} = \frac{1.6}{1} = \boxed{1.6 \text{ oz/min}}$

b) mean value Theorem $x=2$ to $x=4$

$\frac{C(4) - C(2)}{4-2} = \frac{12.8 - 8.8}{2} = \frac{4}{2} = 2$

closed interval

By MVT, since C is differenuable thus continuous and $C'(t)=2$

c) 

midpoint sum

$2(5.3) + 2(11.2) + 2(13.8) = \frac{60.6}{6} = \boxed{10.1 \text{ oz}}$

$\boxed{\text{average amount of coffee in the cup over the time interval } 0 \le t \le 6}$

d) $B'(t) = -16e^{-0.4t} \cdot (-0.4) = 6.4e^{-0.4t}$

$= 6.4e^{-0.4(5)} = 6.4e^{-2} = \boxed{\frac{6.4}{e^2} \text{ oz/min}}$

AP® CALCULUS BC
2013 SCORING GUIDELINES

Question 3

t (minutes)	0	1	2	3	4	5	6
$C(t)$ (ounces)	0	5.3	8.8	11.2	12.8	13.8	14.5

Hot water is dripping through a coffeemaker, filling a large cup with coffee. The amount of coffee in the cup at time t, $0 \le t \le 6$, is given by a differentiable function C, where t is measured in minutes. Selected values of $C(t)$, measured in ounces, are given in the table above.

(a) Use the data in the table to approximate $C'(3.5)$. Show the computations that lead to your answer, and indicate units of measure.

(b) Is there a time t, $2 \le t \le 4$, at which $C'(t) = 2$? Justify your answer.

(c) Use a midpoint sum with three subintervals of equal length indicated by the data in the table to approximate the value of $\frac{1}{6}\int_0^6 C(t)\,dt$. Using correct units, explain the meaning of $\frac{1}{6}\int_0^6 C(t)\,dt$ in the context of the problem.

(d) The amount of coffee in the cup, in ounces, is modeled by $B(t) = 16 - 16e^{-0.4t}$. Using this model, find the rate at which the amount of coffee in the cup is changing when $t = 5$.

(a) $C'(3.5) \approx \dfrac{C(4) - C(3)}{4 - 3} = \dfrac{12.8 - 11.2}{1} = 1.6$ ounces/min

$2 : \begin{cases} 1 : \text{approximation} \\ 1 : \text{units} \end{cases}$

(b) C is differentiable $\Rightarrow$ C is continuous (on the closed interval)

$\dfrac{C(4) - C(2)}{4 - 2} = \dfrac{12.8 - 8.8}{2} = 2$

Therefore, by the Mean Value Theorem, there is at least one time t, $2 < t < 4$, for which $C'(t) = 2$.

$2 : \begin{cases} 1 : \dfrac{C(4) - C(2)}{4 - 2} \\ 1 : \text{conclusion, using MVT} \end{cases}$

(c) $\frac{1}{6}\int_0^6 C(t)\,dt \approx \frac{1}{6}[2 \cdot C(1) + 2 \cdot C(3) + 2 \cdot C(5)]$

$= \frac{1}{6}(2 \cdot 5.3 + 2 \cdot 11.2 + 2 \cdot 13.8)$

$= \frac{1}{6}(60.6) = 10.1$ ounces

$\frac{1}{6}\int_0^6 C(t)\,dt$ is the average amount of coffee in the cup, in ounces, over the time interval $0 \le t \le 6$ minutes.

$3 : \begin{cases} 1 : \text{midpoint sum} \\ 1 : \text{approximation} \\ 1 : \text{interpretation} \end{cases}$

(d) $B'(t) = -16(-0.4)e^{-0.4t} = 6.4e^{-0.4t}$

$B'(5) = 6.4e^{-0.4(5)} = \dfrac{6.4}{e^2}$ ounces/min

$2 : \begin{cases} 1 : B'(t) \\ 1 : B'(5) \end{cases}$

a) $c'(3.5)$ Find the slope $\frac{y_2-y_1}{x_2-x_1}$

$(3, 11.2)$ and $(4, 12.8)$

$$c'(3.5) = \frac{12.8 - 11.2}{4-3} = \frac{1.6}{1} = \boxed{1.6} \text{ oz/min}$$

b) $c'(t) = 2$ on $[\overset{a}{2}, \overset{b}{4}]$

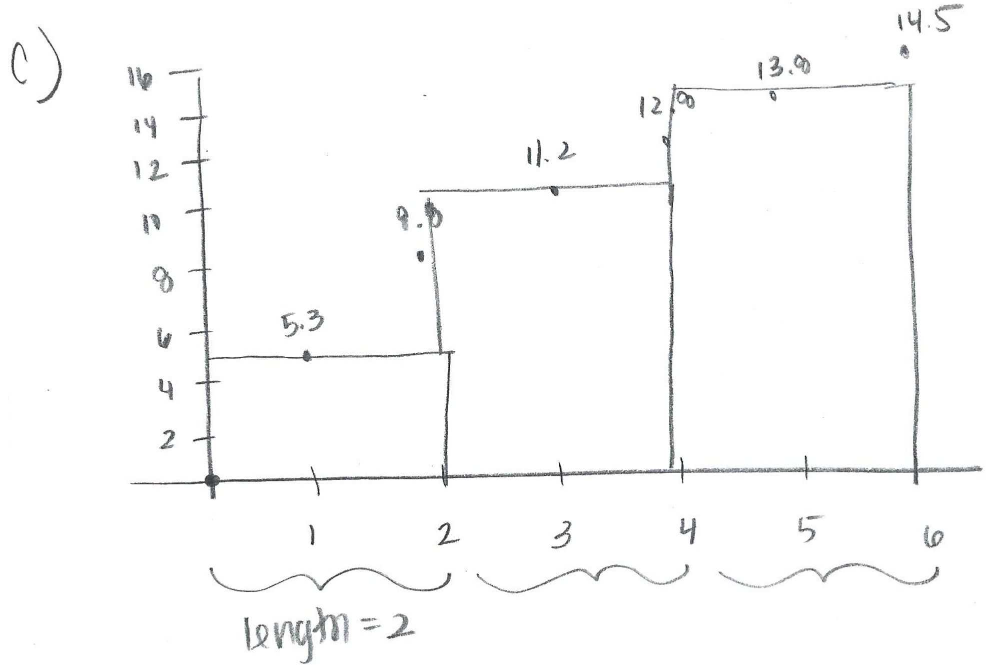
mean value theorem $\quad f'(c) = \frac{f(b) - f(a)}{b-a}$

$$c'(t) = \frac{C(4) - C(2)}{4-2}$$

$$c'(t) = \frac{12.8 - 8.8}{2} = \frac{4}{2} = 2$$

yes, $c'(t) = 2$, by the MVT

c)



← 3 equal subintervals !

midpoint $= 2(5.3) + 2(11.2) + 2(13.8) = 60.6$

we want $\frac{1}{6}\int_0^6 C(t) = \frac{1}{6}(60.6) = \boxed{10.1 \text{ oz}}$

The average amount of coffee in the cup is 10.1 oz over the time interval $0 \le t \le 6$.

d) $B(t) = 16 - 16e^{-0.4t}$

$B'(t) = -16(-0.4)e^{-0.4t}$

$B'(t) = 6.4e^{-0.4t}$

$B'(5) = 6.4e^{-0.4(5)} = 6.4e^{-2} = \boxed{\frac{6.4}{e^2} \text{ oz/min}}$

TABULAR DATA AND REIMANN SUM

2011 AP® CALCULUS BC FREE-RESPONSE QUESTIONS

t (minutes)	0	2	5	9	10
$H(t)$ (degrees Celsius)	66	60	52	44	43

2. As a pot of tea cools, the temperature of the tea is modeled by a differentiable function H for $0 \le t \le 10$, where time t is measured in minutes and temperature $H(t)$ is measured in degrees Celsius. Values of $H(t)$ at selected values of time t are shown in the table above.

   (a) Use the data in the table to approximate the rate at which the temperature of the tea is changing at time $t = 3.5$. Show the computations that lead to your answer.

   (b) Using correct units, explain the meaning of $\frac{1}{10}\int_0^{10} H(t)\,dt$ in the context of this problem. Use a trapezoidal sum with the four subintervals indicated by the table to estimate $\frac{1}{10}\int_0^{10} H(t)\,dt$.

   (c) Evaluate $\int_0^{10} H'(t)\,dt$. Using correct units, explain the meaning of the expression in the context of this problem.

   (d) At time $t = 0$, biscuits with temperature 100°C were removed from an oven. The temperature of the biscuits at time t is modeled by a differentiable function B for which it is known that $B'(t) = -13.84e^{-0.173t}$. Using the given models, at time $t = 10$, how much cooler are the biscuits than the tea?

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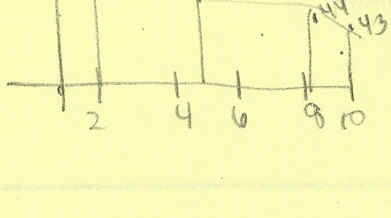
a) rate at $t = 3.5 \Rightarrow$ slope at 3.5

Use table

t	2	5
$H(t)$	60	52

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{52-60}{5-2} = \frac{-8}{3} = \boxed{-2.6666 \text{ or } -2.667 \text{ degrees C per min}}$$

b) $$\int_0^{10} = \frac{1}{2}(2)(66+60) + \frac{1}{2}(3)(60+52) + \frac{1}{2}(4)(52+44)$$
$$+ \frac{1}{2}(1)(44+43) = \frac{529.5}{10} = \boxed{52.95}$$



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2011 SCORING GUIDELINES

Question 2

t (minutes)	0	2	5	9	10
$H(t)$ (degrees Celsius)	66	60	52	44	43

As a pot of tea cools, the temperature of the tea is modeled by a differentiable function H for $0 \le t \le 10$, where time t is measured in minutes and temperature $H(t)$ is measured in degrees Celsius. Values of $H(t)$ at selected values of time t are shown in the table above.

(a) Use the data in the table to approximate the rate at which the temperature of the tea is changing at time $t = 3.5$. Show the computations that lead to your answer.

(b) Using correct units, explain the meaning of $\frac{1}{10}\int_0^{10} H(t)\,dt$ in the context of this problem. Use a trapezoidal sum with the four subintervals indicated by the table to estimate $\frac{1}{10}\int_0^{10} H(t)\,dt$.

(c) Evaluate $\int_0^{10} H'(t)\,dt$. Using correct units, explain the meaning of the expression in the context of this problem.

(d) At time $t = 0$, biscuits with temperature 100°C were removed from an oven. The temperature of the biscuits at time t is modeled by a differentiable function B for which it is known that $B'(t) = -13.84e^{-0.173t}$. Using the given models, at time $t = 10$, how much cooler are the biscuits than the tea?

(a) $H'(3.5) \approx \dfrac{H(5) - H(2)}{5 - 2}$

$= \boxed{\dfrac{52 - 60}{3}} = -2.666$ or -2.667 degrees Celsius per minute

1 : answer

(b) $\frac{1}{10}\int_0^{10} H(t)\,dt$ is the average temperature of the tea, in degrees Celsius, over the 10 minutes.

$$\frac{1}{10}\int_0^{10} H(t)\,dt \approx \frac{1}{10}\left(2\cdot\frac{66+60}{2} + 3\cdot\frac{60+52}{2} + 4\cdot\frac{52+44}{2} + 1\cdot\frac{44+43}{2}\right)$$
$$= 52.95$$

3 : { 1 : meaning of expression; 1 : trapezoidal sum; 1 : estimate }

(c) $\int_0^{10} H'(t)\,dt = H(10) - H(0) = 43 - 66 = -23$

The temperature of the tea drops 23 degrees Celsius from time $t = 0$ to time $t = 10$ minutes.

2 : { 1 : value of integral; 1 : meaning of expression }

(d) $B(10) = 100 + \int_0^{10} B'(t)\,dt = 34.18275$; $\quad H(10) - B(10) = 8.817$

The biscuits are 8.817 degrees Celsius cooler than the tea.

3 : { 1 : integrand; 1 : uses $B(0) = 100$; 1 : answer }

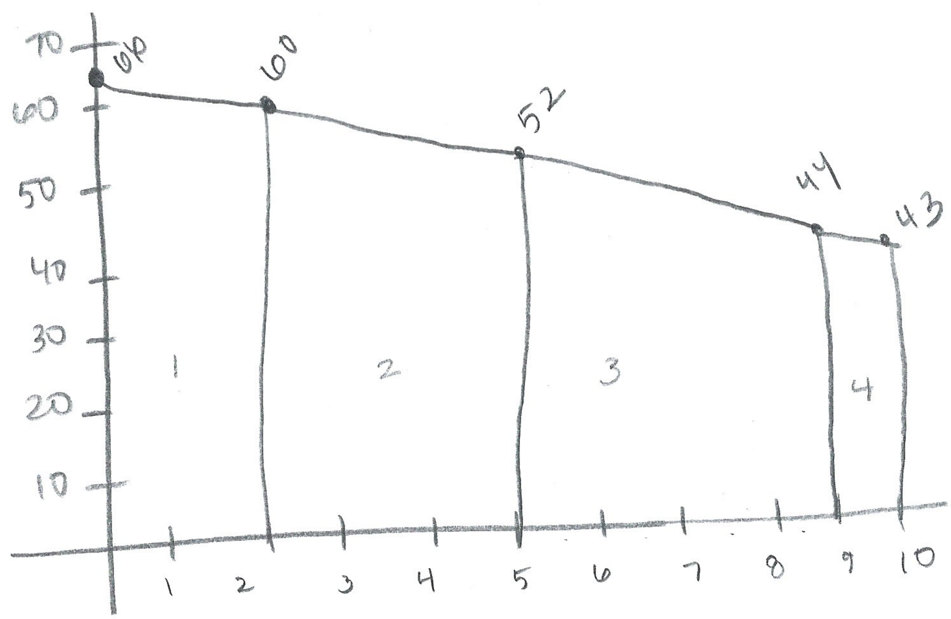
a) rate of change at $t = 3.5$

Find $H'(3.5) \rightarrow$ slope $\dfrac{y_2 - y_1}{x_2 - x_1}$

$(2, 60)$ and $(5, 52)$

$$H'(3.5) = \frac{52-60}{5-2} = \frac{-8}{3} = \boxed{-2.667\ {}^\circ C/min}$$

b)



4 subintervals

$$\int_0^{10} H(t) = \frac{1}{2}(2)(66+60) + \frac{1}{2}(3)(60+52)$$

$$+ \frac{1}{2}(4)(52+44) + \frac{1}{2}(1)(44+43)$$

$$= 529.5$$

$$\frac{1}{10}\int_0^{10} H(t) = \frac{1}{10}(529.5) = \boxed{52.95}$$

the average temperature of the tea is $52.95\ {}^\circ C$ over the time interval $0 \le t \le 10$.

c) $\int_0^{10} H'(t) = H(t)\Big|_0^{10} = H(10) - H(10)$

$= 43 - 66 = \boxed{-23}$

The temperature of tea drops 23°C from $t=0$ to time $t=10$.

d) $B'(t) = -13.84e^{-0.173t}$ $\qquad B(0) = 100$

$B(10) = 100 + \int_0^{10} B'(t)\,dt$ — do on calc.

starting temp

as temp. cools over 10min

$= 100 + -65.8172$

$= 34.1828\ °C$

Tea at 10min = 43°C (chart)

Biscuit at 10min = 34.1828 °C

The biscuit is cooler by 8.8172 °C than the tea.

x	2	3	5	8	13
$f(x)$	1	4	−2	3	6

5. Let f be a function that is twice differentiable for all real numbers. The table above gives values of f for selected points in the closed interval $2 \le x \le 13$.

(a) Estimate $f'(4)$. Show the work that leads to your answer.

(b) Evaluate $\int_2^{13} (3 - 5f'(x))\,dx$. Show the work that leads to your answer.

(c) Use a left Riemann sum with subintervals indicated by the data in the table to approximate $\int_2^{13} f(x)\,dx$. Show the work that leads to your answer.

(d) Suppose $f'(5) = 3$ and $f''(x) < 0$ for all x in the closed interval $5 \le x \le 8$. Use the line tangent to the graph of f at $x = 5$ to show that $f(7) \le 4$. Use the secant line for the graph of f on $5 \le x \le 8$ to show that $f(7) \ge \frac{4}{3}$.

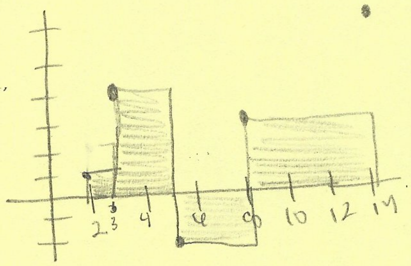
a) $f'(4) \rightarrow$ slope at 4

use table

x	3	5
$f(x)$	4	-2

$\frac{y_2-y_1}{x_2-x_1} = \frac{-2-4}{5-3} = \frac{-6}{2} = \boxed{-3}$

c)



left reimann sum

$(1)(1) + 4(2) + -2(3)$

$+5(3) = 1+8-6+15 = \boxed{18}$

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2009 SCORING GUIDELINES

Question 5

x	2	3	5	8	13
$f(x)$	1	4	−2	3	6

Let f be a function that is twice differentiable for all real numbers. The table above gives values of f for selected points in the closed interval $2 \le x \le 13$.

(a) Estimate $f'(4)$. Show the work that leads to your answer.

(b) Evaluate $\int_2^{13} (3 - 5f'(x))\,dx$. Show the work that leads to your answer.

(c) Use a left Riemann sum with subintervals indicated by the data in the table to approximate $\int_2^{13} f(x)\,dx$. Show the work that leads to your answer.

(d) Suppose $f'(5) = 3$ and $f''(x) < 0$ for all x in the closed interval $5 \le x \le 8$. Use the line tangent to the graph of f at $x = 5$ to show that $f(7) \le 4$. Use the secant line for the graph of f on $5 \le x \le 8$ to show that $f(7) \ge \frac{4}{3}$.

(a) $f'(4) \approx \dfrac{f(5) - f(3)}{5 - 3} = -3$

1 : answer

(b) $\int_2^{13} (3 - 5f'(x))\,dx = \int_2^{13} 3\,dx - 5\int_2^{13} f'(x)\,dx$
$= 3(13 - 2) - 5(f(13) - f(2)) = 8$

2 : { 1 : uses Fundamental Theorem of Calculus; 1 : answer }

(c) $\int_2^{13} f(x)\,dx \approx f(2)(3 - 2) + f(3)(5 - 3)$
$+ f(5)(8 - 5) + f(8)(13 - 8) = 18$

2 : { 1 : left Riemann sum; 1 : answer }

(d) An equation for the tangent line is $y = -2 + 3(x - 5)$. Since $f''(x) < 0$ for all x in the interval $5 \le x \le 8$, the line tangent to the graph of $y = f(x)$ at $x = 5$ lies above the graph for all x in the interval $5 < x \le 8$.

Therefore, $f(7) \le -2 + 3 \cdot 2 = 4$.

An equation for the secant line is $y = -2 + \frac{5}{3}(x - 5)$.

Since $f''(x) < 0$ for all x in the interval $5 \le x \le 8$, the secant line connecting $(5, f(5))$ and $(8, f(8))$ lies below the graph of $y = f(x)$ for all x in the interval $5 < x < 8$.

Therefore, $f(7) \ge -2 + \frac{5}{3} \cdot 2 = \frac{4}{3}$.

4 : { 1 : tangent line; 1 : shows $f(7) \le 4$; 1 : secant line; 1 : shows $f(7) \ge \frac{4}{3}$ }

a) $f'(4) \rightarrow$ slope $\dfrac{y_2-y_1}{x_2-x_1}$

$(3,4)$ and $(5,-2)$

$$f'(4) = \frac{-2-4}{5-3} = \frac{-6}{2} = \boxed{-3}$$

b) $\displaystyle\int_2^{13} (3 - 5f'(x))\,dx$

$$\int_2^{13} 3 - \int_2^{13} 5f'(x)$$

$$3x - 5f(x) \Big|_2^{13}$$

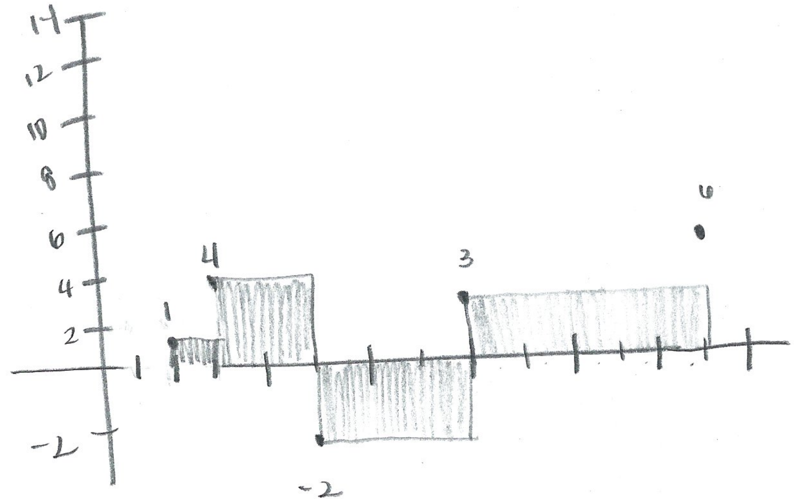
$$= [3(13) - 5f(13)] - [3(2) - 5f(2)]$$

$$= [39 - 5(6)] - [6 - 5(1)]$$

$$= [39 - 30] - [6 - 5]$$

$$= 9 - 1 = \boxed{8}$$

c)



$$\int_2^{13} f(x) = 1(1) + 2(4) - 3(2) + 5(3)$$

$$= 1 + 8 - 6 + 15 = \boxed{18}$$

d) $x = 5$ then $f(5) = -2 \Rightarrow$ point $(5, -2)$

we know $f'(5) = 3 \leftarrow$ slope at $x = 5$

tangent line $\quad y - y_1 = m(x - x_1)$

$$y + +2 = 3(x - 5)$$

$$y + \cancel{2} = 3x - 15$$
$$\quad -\cancel{2} \qquad\quad -2$$

$$y = 3x - 17$$

$$f(x) = 3x - 17$$

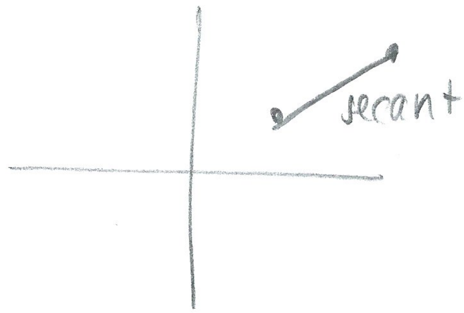
$$f(7) = 3(7) - 17$$
$$= 21 - 17$$

$f(7) = 4$ $\quad \boxed{\text{Thus } f(7) \leq 4}$ ✓

secant line from $5 \le x \le 8$

(5, −2) and (8, 3)

*Find y-values from table



$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 + +2}{8 - 5} = \frac{5}{3}$ and point $(5, -2)$

$y - y_1 = m(x - x_1)$

$y + +2 = \frac{5}{3}(x - 5)$

$y + \cancel{2} = \frac{5}{3}x - \frac{25}{3}$ $\qquad -\frac{25}{3}$

$-\cancel{2}$

$\frac{-2 \cdot 3}{1 \cdot 3}$ $\qquad \frac{-6}{3}$

$y = \frac{5x}{3} - \frac{31}{3}$

$f(x) = \frac{5x}{3} - \frac{31}{3}$

$f(7) = \frac{5(7)}{3} - \frac{31}{3} = \frac{35}{3} - \frac{31}{3} = \frac{4}{3} \ge \frac{4}{3}$ ✓

$\boxed{\text{Thus } f(7) \ge \frac{4}{3}}$