Ratio Test

9.6 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Verifying a Formula In Exercises 1–4, verify the formula.

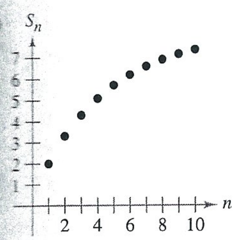
1. $\dfrac{(n+1)!}{(n-2)!} = (n+1)(n)(n-1)$

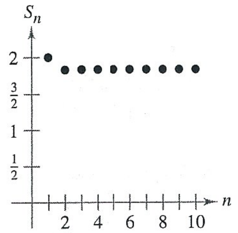
2. $\dfrac{(2k-2)!}{(2k)!} = \dfrac{1}{(2k)(2k-1)}$

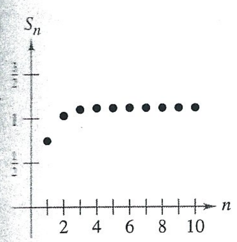
3. $1 \cdot 3 \cdot 5 \cdots (2k-1) = \dfrac{(2k)!}{2^k k!}$

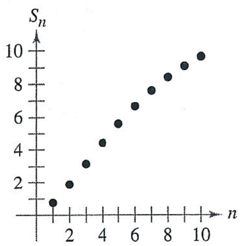
4. $\dfrac{1}{1 \cdot 3 \cdot 5 \cdots (2k-5)} = \dfrac{2^k k!(2k-3)(2k-1)}{(2k)!}, \quad k \geq 3$

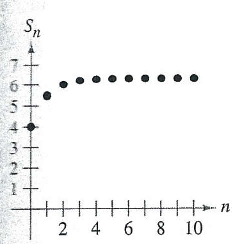
Matching In Exercises 5–10, match the series with the graph of its sequence of partial sums. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

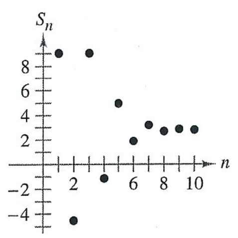
(a)


(b)


(c)


(d)


(e)


(f)


5. $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{3}{4}\right)^n$

6. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n\left(\frac{1}{n!}\right)$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{n!}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}4}{(2n)!}$

9. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{4n}{5n-3}\right)^n$

10. $\displaystyle\sum_{n=0}^{\infty} 4e^{-n}$

Numerical, Graphical, and Analytic Analysis In Exercises 11 and 12, (a) verify that the series converges, (b) use a graphing utility to find the indicated partial sum S_n and complete the table, (c) use a graphing utility to graph the first 10 terms of the sequence of partial sums, (d) use the table to estimate the sum of the series, and (e) explain the relationship between the magnitudes of the terms of the series and the rate at which the sequence of partial sums approaches the sum of the series.

n	5	10	15	20	25
S_n					

11. $\displaystyle\sum_{n=1}^{\infty} n^3\left(\frac{1}{2}\right)^n$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{n!}$

Using the Ratio Test In Exercises 13–34, use the Ratio Test to determine the convergence or divergence of the series.

13. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{5^n}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$

15. $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{3^n}$

16. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!}$

17. $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{6}{5}\right)^n$

18. $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{7}{8}\right)^n$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{4^n}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{n^4}$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{3^n}$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+2)}{n(n+1)}$

23. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!}$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(3/2)^n}{n^2}$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n3^n}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n^5}$

27. $\displaystyle\sum_{n=0}^{\infty} \frac{e^n}{n!}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

29. $\displaystyle\sum_{n=0}^{\infty} \frac{6^n}{(n+1)^n}$

30. $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^2}{(3n)!}$

31. $\displaystyle\sum_{n=0}^{\infty} \frac{5^n}{2^n+1}$

32. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^{4n}}{(2n+1)!}$

33. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1} n!}{1 \cdot 3 \cdot 5 \cdots (2n+1)}$

34. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n[2 \cdot 4 \cdot 6 \cdots (2n)]}{2 \cdot 5 \cdot 8 \cdots (3n-1)}$

Ratio Test two

(13) $\sum_{n=1}^{\infty} \frac{1}{5^n} \Rightarrow \lim_{n\to\infty} \frac{\frac{1}{5^{n+1}}}{\frac{1}{5^n}} = \lim_{n\to\infty} \frac{1}{5^{n+1}} \cdot \frac{5^n}{1} = \lim_{n\to\infty} \frac{5^n}{5^{n+1}}$

$= \lim_{n\to\infty} \frac{\cancel{5^n}}{\cancel{5^n} \cdot 5^1} = \lim_{n\to\infty} \frac{1}{5} = \frac{1}{5} < 1$

$\boxed{\text{converges}}$

(14) $\sum_{n=1}^{\infty} \frac{1}{n!} \Rightarrow \lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n\to\infty} \frac{1}{(n+1)!} \cdot \frac{n!}{1} = \lim_{n\to\infty} \frac{n!}{(n+1)!}$

$= \lim_{n\to\infty} \frac{\cancel{n \cdots 3 \cdot 2 \cdot 1}}{(n+1) \cdot \cancel{n \cdots 3 \cdot 2 \cdot 1}} =$

$= \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$ $\boxed{\text{converges}}$

(15) $\sum_{n=0}^{\infty} \frac{n!}{3^n} \Rightarrow \lim_{n\to\infty} \frac{\frac{(n+1)!}{3^{n+1}}}{\frac{n!}{3^n}} = \lim_{n\to\infty} \frac{(n+1)!}{3^{n+1}} \cdot \frac{3^n}{n!}$

$= \lim_{n\to\infty} \frac{(n+1) \cdot \cancel{n \cdots 3 \cdot 2 \cdot 1}}{\cancel{3^n} \cdot 3^1} \cdot \frac{\cancel{3^n}}{\cancel{n \cdots 3 \cdot 2 \cdot 1}}$

$= \lim_{n\to\infty} \frac{n+1}{3} = \infty > 1$ $\boxed{\text{diverges}}$

(16) $\sum_{n=1}^{\infty} \frac{2^n}{n!} \Rightarrow \lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}$

$= \lim_{n\to\infty} \frac{\cancel{2^n}\cdot 2^1}{(n+1)\cdot \cancel{n \cdots 3\cdot 2\cdot 1}} \cdot \frac{\cancel{n \cdots 3\cdot 2\cdot 1}}{\cancel{2^n}}$

$= \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1$ $\boxed{\text{converges}}$

(17) $\sum_{n=1}^{\infty} n\left(\frac{6}{5}\right)^n \Rightarrow \lim_{n\to\infty} \frac{(n+1)\left(\frac{6}{5}\right)^{n+1}}{n\left(\frac{6}{5}\right)^n} = \frac{(n+1)\cdot\cancel{\left(\frac{6}{5}\right)^n}\cdot\left(\frac{6}{5}\right)^1}{\frac{n}{1}\cancel{\left(\frac{6}{5}\right)^n}} \frac{1}{n}$

$= \lim_{n\to\infty} \frac{6n+6}{5}\cdot\frac{1}{n} = \lim_{n\to\infty} \frac{6n+6}{5n} = \frac{6}{5} > 1$

$\boxed{\text{Diverges}}$

(18) $\sum_{n=1}^{\infty} n\left(\frac{7}{8}\right)^n \Rightarrow \lim_{n\to\infty} \frac{(n+1)\left(\frac{7}{8}\right)^{n+1}}{n\left(\frac{7}{8}\right)^n} = \frac{(n+1)\cancel{\left(\frac{7}{8}\right)^n}\cdot\left(\frac{7}{8}\right)^1}{\frac{n}{1}\cancel{\left(\frac{7}{8}\right)^n}}$

$= \lim_{n\to\infty} \frac{7n+7}{8}\cdot\frac{1}{n}$

$= \lim_{n\to\infty} \frac{7n+7}{8n} = \frac{7}{8} < 1$ $\boxed{\text{converges}}$

(19) $\sum_{n=1}^{\infty} \frac{n}{4^n} \Rightarrow \lim_{n\to\infty} \frac{\frac{n+1}{4^{n+1}}}{\frac{n}{4^n}} = \lim_{n\to\infty} \frac{n+1}{4^{n+1}} \cdot \frac{4^n}{n}$

$= \lim_{n\to\infty} \frac{n+1}{\cancel{4^n} \cdot 4} \cdot \frac{\cancel{4^n}}{n} = \lim_{n\to\infty} \frac{n+1}{4n} = \frac{1}{4} < 1$ $\boxed{\text{converges}}$

(20) $\sum_{n=1}^{\infty} \frac{5^n}{n^4} \Rightarrow \lim_{n\to\infty} \frac{\frac{5^{n+1}}{(n+1)^4}}{\frac{5^n}{n^4}} = \lim_{n\to\infty} \frac{\cancel{5^n} \cdot 5}{(n+1)^4} \cdot \frac{n^4}{\cancel{5^n}}$

$= \lim_{n\to\infty} \frac{5n^4}{(n+1)^4} = 5 > 1$ $\boxed{\text{Diverges}}$

(21) $\sum_{n=1}^{\infty} \frac{n^3}{3^n} \Rightarrow \lim_{n\to\infty} \frac{\frac{(n+1)^3}{3^{n+1}}}{\frac{n^3}{3^n}} = \lim_{n\to\infty} \frac{(n+1)^3}{\cancel{3^n} \cdot 3^1} \cdot \frac{\cancel{3^n}}{n^3}$

$= \lim_{n\to\infty} \frac{(n+1)^3}{3n^3} = \frac{1}{3} < 1$ $\boxed{\text{converges}}$

(22) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+2)}{n(n+1)} \Rightarrow \lim_{n\to\infty} \frac{(-1)^{n+1+1}(n+1+2)}{(n+1)(n+1+1)}$

$= \lim_{n\to\infty} \frac{(-1)^{n+2}(n+3)}{(n+1)(n+2)}$

$= \lim \frac{(-1)^n \cdot (-1)^2 (n+3)}{n^2+3n+2} = 0 < 1$ $\boxed{\text{converges}}$

(23) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n!} \Rightarrow \lim_{n\to\infty} \frac{(-1)^{n+1}\cdot 2^{n+1}}{(n+1)!} \cdot \frac{n!}{(-1)^n \cdot 2^n}$

$$= \lim_{n\to\infty} \frac{\cancel{(-1)^n}\cdot(-1)\cdot \cancel{2^n}\cdot 2^1}{(n+1)\cdot \cancel{n \cdots 3\cdot 2\cdot 1}} \cdot \frac{\cancel{n\cdots 3\cdot 2\cdot 1}}{\cancel{(-1)^n}\cancel{2^n}}$$

$$= \lim_{n\to\infty} \frac{-2}{n+1} = 0 < 1 \quad \boxed{\text{converges}}$$

(24) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\left(\frac{3}{2}\right)^n}{n^2} \Rightarrow \lim_{n\to\infty} \frac{(-1)^{n+1-1}\left(\frac{3}{2}\right)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(-1)^{n-1}\left(\frac{3}{2}\right)^n}$

$$= \lim_{n\to\infty} \frac{\cancel{(-1)^n}\cancel{\left(\frac{3}{2}\right)^n}\cdot\left(\frac{3}{2}\right)}{(n+1)^2} \cdot \frac{n^2}{\cancel{(-1)^n}\cdot(-1)^{-1}\cancel{\left(\frac{3}{2}\right)^n}}$$

$$= \lim_{n\to\infty} -\frac{3n^2}{2} \cdot \frac{1}{(n+1)^2}$$

n^2+2n+1

$$= \lim_{n\to\infty} \frac{-3n^2}{2n^2+4n+2} = -\frac{3}{2} < 1 \quad \boxed{\text{converges}}$$

(25) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n3^n} \Rightarrow \lim_{n\to\infty} \frac{(n+1)!}{(n+1)\,3^{n+1}} \cdot \frac{n\cdot 3^n}{n!}$

$$= \lim_{n\to\infty} \frac{\cancel{(n+1)}\cdot\cancel{n\cdots 3\cdot 2\cdot 1}}{\cancel{(n+1)}\cdot\cancel{3^n}\cdot 3^1} \cdot \frac{n\cdot\cancel{3^n}}{\cancel{n\cdots 3\cdot 2\cdot 1}}$$

$$= \lim_{n\to\infty} \frac{n}{3} = \infty > 1 \quad \boxed{\text{diverges}}$$

(26) $\sum_{n=1}^{\infty} \frac{(2n)!}{n^5} \Rightarrow \lim_{n\to\infty} \frac{2(n+1)!}{(n+1)^5} \cdot \frac{n^5}{(2n)!}$

$= \lim_{n\to\infty} \frac{(2n+2)!}{(n+1)^5} \cdot \frac{n^5}{(2n)!}$

$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)\cancel{(2n)\cdots 3\cdot 2\cdot 1}}{(n+1)^5} \cdot \frac{n^5}{\cancel{2n\cdots 3\cdot 2\cdot 1}}$

$= \lim_{n\to\infty} \frac{4n^7}{n^5} = \infty > 1$ $\boxed{\text{Diverges}}$

(27) $\sum_{n=0}^{\infty} \frac{e^n}{n!} \Rightarrow \lim_{n\to\infty} \frac{e^{n+1}}{(n+1)!} \cdot \frac{n!}{e^n}$

$= \lim_{n\to\infty} \frac{\cancel{e^n}\cdot e^1}{(n+1)\cancel{n!}} \cdot \frac{\cancel{n!}}{\cancel{e^n}} = \lim_{n\to\infty} \frac{e}{n+1} = 0 < 1$ $\boxed{\text{converges}}$

(28) $\sum_{n=1}^{\infty} \frac{n!}{n^n} \Rightarrow \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}$

$= \lim_{n\to\infty} \frac{\cancel{(n+1)}\cancel{n!}}{(n+1)^n \cdot \cancel{(n+1)}} \cdot \frac{n^n}{\cancel{n!}}$

$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = e^{-1} = \frac{1}{e} < 1$ $\boxed{\text{converges}}$

$= \lim_{n\to\infty} 1^{\infty}$ indeterminat form

$=$

$n \ln\left(\frac{n}{n+1}\right) = \frac{\ln \frac{n}{n+1}}{\frac{1}{n}}$

$\frac{\frac{1}{\left(\frac{n}{n+1}\right)} \cdot \frac{(n+1)\cdot 1 - n\cdot 1}{(n+1)^2}}{-1n^{-2} \quad \frac{-1}{n^2}}$

$\frac{n+1}{n} \cdot \frac{1}{(n+1)^2} = \frac{1}{n(n+1)} \div \frac{n^2}{1}$

$= \frac{-n}{n+1} = -1$ $\boxed{e^{-1}}$