SERIES

2012 AP® CALCULUS BC FREE-RESPONSE QUESTIONS

6. The function g has derivatives of all orders, and the Maclaurin series for g is

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+3} = \frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7} - \cdots.$$

(a) Using the ratio test, determine the interval of convergence of the Maclaurin series for g.

(b) The Maclaurin series for g evaluated at $x = \frac{1}{2}$ is an alternating series whose terms decrease in absolute value to 0. The approximation for $g\left(\frac{1}{2}\right)$ using the first two nonzero terms of this series is $\frac{17}{120}$. Show that this approximation differs from $g\left(\frac{1}{2}\right)$ by less than $\frac{1}{200}$.

(c) Write the first three nonzero terms and the general term of the Maclaurin series for $g'(x)$.

$2n+3$

$2n+2+1$

a) $$\lim_{n\to\infty}\left|\frac{x^{2(n+1)+1}}{\underset{2n+2+3}{2(n+1)+3}} \cdot \frac{2n+3}{x^{2n+1}}\right| = \lim_{n\to\infty}\left|\frac{\cancel{x^{2n+1}}x^2}{2n+5}\cdot\frac{2n+3}{\cancel{x^{2n+1}}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{2n+3}{2n+5}\cdot x^2\right| = 1\,|x^2|<1$$

$$\sqrt{|x^2|} < \sqrt{1}$$

$$|x|<1$$

$$-1 \le x \le 1$$

look at endpoints

$$x=1 \quad \frac{(-1)^n 1^{2n+1}}{2n+3} = \frac{1}{2n+3} = \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9}$$

alt. series
① decreasing terms
② $\lim_{n\to\infty} f(x) = 0$
converges

$$\text{try } x=-1 \quad \frac{(-1)^n(-1)^{2n+1}}{2n+3} = \frac{(-1)^{3n+1}}{2n+3} = -\frac{1}{3} + \frac{1}{5} - \frac{1}{7}$$

alt. series
① decreasing terms
② $\lim_{n\to\infty} f(x) = 0$
converges

series converges $\boxed{-1 \le x \le 1}$

AP® CALCULUS BC
2012 SCORING GUIDELINES

Question 6

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$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+3} = \frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7} - \cdots$.

(a) Using the ratio test, determine the interval of convergence of the Maclaurin series for g.

(b) The Maclaurin series for g evaluated at $x = \frac{1}{2}$ is an alternating series whose terms decrease in absolute value to 0. The approximation for $g\left(\frac{1}{2}\right)$ using the first two nonzero terms of this series is $\frac{17}{120}$. Show that this approximation differs from $g\left(\frac{1}{2}\right)$ by less than $\frac{1}{200}$.

(c) Write the first three nonzero terms and the general term of the Maclaurin series for $g'(x)$.

(a) $\left|\frac{x^{2n+3}}{2n+5} \cdot \frac{2n+3}{x^{2n+1}}\right| = \left(\frac{2n+3}{2n+5}\right) \cdot x^2$

$\lim_{n\to\infty}\left(\frac{2n+3}{2n+5}\right) \cdot x^2 = x^2$

$x^2 < 1 \Rightarrow -1 < x < 1$
The series converges when $-1 < x < 1$.

When $x = -1$, the series is $-\frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$
This series converges by the Alternating Series Test.

When $x = 1$, the series is $\frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \cdots$
This series converges by the Alternating Series Test.

Therefore, the interval of convergence is $-1 \le x \le 1$.

$5: \begin{cases} 1: \text{sets up ratio} \\ 1: \text{computes limit of ratio} \\ 1: \text{identifies interior of interval of convergence} \\ 1: \text{considers both endpoints} \\ 1: \text{analysis and interval of convergence} \end{cases}$

(b) $\left|g\left(\frac{1}{2}\right) - \frac{17}{120}\right| < \frac{\left(\frac{1}{2}\right)^5}{7} = \frac{1}{224} < \frac{1}{200}$

$2: \begin{cases} 1: \text{uses the third term as an error bound} \\ 1: \text{error bound} \end{cases}$

(c) $g'(x) = \frac{1}{3} - \frac{3}{5}x^2 + \frac{5}{7}x^4 + \cdots + (-1)^n\left(\frac{2n+1}{2n+3}\right)x^{2n} + \cdots$

$2: \begin{cases} 1: \text{first three terms} \\ 1: \text{general term} \end{cases}$

b) $R_n(x) = \dfrac{f^{n+1}(z)(x-c)^{n+1}}{(n+1)!}$

$x = \frac{1}{2}$
$c = 0$
$n = 2$
$z =$

$R_2(x) = \dfrac{f^3(z)(x-0)^3}{3!}$

$R_2(x) = \dfrac{f^3(z)\,x^3}{3!}$ 3rd term of series

first 2 terms

$\underbrace{\frac{x}{3} - \frac{x^3}{5}} + \frac{x^5}{7}$ 3rd term

$\downarrow$

$R_2(\tfrac{1}{2}) = \dfrac{x^5}{7}$

$R_2(\tfrac{1}{2}) = \dfrac{(\frac{1}{2})^5}{7} = \dfrac{\frac{1}{32}}{\frac{7}{1}} = \frac{1}{32} \cdot \frac{1}{7} = \boxed{\frac{1}{224} < \frac{1}{200}}$

$\begin{array}{r} 32 \\ \times 7 \\ \hline 224 \end{array}$

$\left| g(\tfrac{1}{2}) - \tfrac{17}{120} \right| <$ 3rd term
error for 2nd term

$\frac{x^5}{7} = \frac{(\frac{1}{2})^5}{7} = \frac{1}{224} < \frac{1}{200}$

c) $g = \frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7} + \dots + (-1)^n \dfrac{x^{2n+1}}{2n+3}$

$\boxed{g'(x) = \frac{1}{3} - \frac{3x^2}{5} + \frac{5x^4}{7} + \dots + (-1)^n \dfrac{(2n+1)\,x^{2n}}{(2n+3)}}$

2013 AP® CALCULUS BC FREE-RESPONSE QUESTIONS

6. A function f has derivatives of all orders at $x = 0$. Let $P_n(x)$ denote the nth-degree Taylor polynomial for f about $x = 0$.

   (a) It is known that $f(0) = -4$ and that $P_1\left(\frac{1}{2}\right) = -3$. Show that $f'(0) = 2$.

   (b) It is known that $f''(0) = -\frac{2}{3}$ and $f'''(0) = \frac{1}{3}$. Find $P_3(x)$.

   (c) The function h has first derivative given by $h'(x) = f(2x)$. It is known that $h(0) = 7$. Find the third-degree Taylor polynomial for h about $x = 0$.

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a) $P_1(x) = f(0) + \frac{f'(0)(x-0)^1}{1!}$

$P_1(x) = f(0) + \frac{f'(0)\,x}{1!} = -4 + \frac{f'(0)\,x}{1!} = -4 + f'(0)\,x$

$P_1\left(\frac{1}{2}\right) = -4 + f'(0)\,\frac{1}{2}$

$-3 = -4 + f'(0)\,\frac{1}{2}$

$+4 \quad +4$

$2 \cdot 1 = f'(0) \cdot \frac{1}{2} \cdot 2$

$\boxed{2 = f'(0)}$

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(b) It is known that $f''(0) = -\frac{2}{3}$ and $f'''(0) = \frac{1}{3}$. Find $P_3(x)$.

(c) The function h has first derivative given by $h'(x) = f(2x)$. It is known that $h(0) = 7$. Find the third-degree Taylor polynomial for h about $x = 0$.

(a) $P_1(x) = f(0) + f'(0)x = -4 + f'(0)x$

$$P_1\left(\frac{1}{2}\right) = -4 + f'(0)\cdot\frac{1}{2} = -3$$

$$f'(0)\cdot\frac{1}{2} = 1$$

$$f'(0) = 2$$

$2 : \begin{cases} 1 : \text{uses } P_1(x) \\ 1 : \text{verifies } f'(0) = 2 \end{cases}$

(b) $P_3(x) = -4 + 2x + \left(-\frac{2}{3}\right)\cdot\frac{x^2}{2!} + \frac{1}{3}\cdot\frac{x^3}{3!}$

$$= -4 + 2x - \frac{1}{3}x^2 + \frac{1}{18}x^3$$

$3 : \begin{cases} 1 : \text{first two terms} \\ 1 : \text{third term} \\ 1 : \text{fourth term} \end{cases}$

(c) Let $Q_n(x)$ denote the Taylor polynomial of degree n for h about $x = 0$.

$$h'(x) = f(2x) \Rightarrow Q_3'(x) = -4 + 2(2x) - \frac{1}{3}(2x)^2$$

$$Q_3(x) = -4x + 4\cdot\frac{x^2}{2} - \frac{4}{3}\cdot\frac{x^3}{3} + C;\; C = Q_3(0) = h(0) = 7$$

$$Q_3(x) = 7 - 4x + 2x^2 - \frac{4}{9}x^3$$

OR

$$h'(x) = f(2x),\; h''(x) = 2f'(2x),\; h'''(x) = 4f''(2x)$$

$$h'(0) = f(0) = -4,\; h''(0) = 2f'(0) = 4,\; h'''(0) = 4f''(0) = -\frac{8}{3}$$

$$Q_3(x) = 7 - 4x + 4\cdot\frac{x^2}{2!} - \frac{8}{3}\cdot\frac{x^3}{3!} = 7 - 4x + 2x^2 - \frac{4}{9}x^3$$

$4 : \begin{cases} 2 : \text{applies } h'(x) = f(2x) \\ 1 : \text{constant term} \\ 1 : \text{remaining terms} \end{cases}$

b) $P_3(x) = f(c) + \frac{f'(c)(x-c)^1}{1!} + \frac{f''(c)(x-c)^2}{2!} + \frac{f'''(c)(x-c)^3}{3!}$

$P_3(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!}$

$P_3(x) = -4 + 2x + \frac{-\cancel{2}x^2}{3 \cdot \cancel{2}!} + \frac{1x^3}{3 \cdot 3!}$ $\quad 3 \cdot 2 \cdot 1$

$$\boxed{P_3(x) = -4 + 2x - \frac{x^2}{3} + \frac{x^3}{18}}$$

c) $h(c) + \frac{h'(c)(x-c)^1}{1!} + \frac{h''(c)(x-c)^2}{2!} + \frac{h'''(c)(x-c)^3}{3!}$

$h'(x) = f(2x)$

$h'(c) = f(2c) = f(0)$

$h''(x) = f'(2x) \cdot 2$

$h''(c) = f'(2c) \cdot 2 = 2f'(0)$

$h'''(x) = f''(2x) \cdot 2 \cdot 2$

$h'''(c) = f''(2c) \cdot 4 = 4f''(0)$

$$h(0) + \frac{f(0)x}{1!} + \frac{2f'(0)x^2}{2!} + \frac{4f''(0)x^3}{3!}$$

$$7 + -4x + 2\cdot\frac{\cancel{2}x^2}{\cancel{2}!} + 4\cdot\frac{-\cancel{2}}{3\cdot 3!}x^3$$

$3\cdot\cancel{2}\cdot 1$

$$\boxed{7 - 4x + 2x^2 - \frac{4}{9}x^3}$$

6. The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n} x^n = x - \frac{3}{2}x^2 + 3x^3 - \cdots + \frac{(-3)^{n-1}}{n} x^n + \cdots$ and converges to $f(x)$ for $|x| < R$, where R is the radius of convergence of the Maclaurin series.

   (a) Use the ratio test to find R.

   (b) Write the first four nonzero terms of the Maclaurin series for f', the derivative of f. Express f' as a rational function for $|x| < R$.

   (c) Write the first four nonzero terms of the Maclaurin series for e^x. Use the Maclaurin series for e^x to write the third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$.

STOP

END OF EXAM

2015 RELEASED FREE RESPONSE SOLUTIONS – MR. CALCULUS

2015 BC #6
(no calculator)

$$\sum_{n=1}^{\infty}\frac{(-3)^{n-1}x^n}{n} = x-\frac{3}{2}x^2+3x^3-\cdots+\frac{(-3)^{n-1}x^n}{n}+\cdots$$

converges to $f(x)$ for $|x|<R$, the radius of convergence

(a)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\dfrac{3^{n+1-1}x^{n+1}}{n+1}}{\dfrac{3^{n-1}x^n}{n}}\right| =$$

$$\lim_{n\to\infty}\left|\frac{3^{n+1-1}x^{n+1}}{n+1}\cdot\frac{n}{3^{n-1}x^n}\right| = \lim_{n\to\infty}\left|\frac{3^n x^n x}{n+1}\cdot\frac{n}{3^n 3^{-1}x^n}\right| = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)|3x| = 3|x|<1 \Rightarrow |x|<\frac{1}{3} \Rightarrow \boxed{R=\frac{1}{3}}$$

(b)

$$f'(x) = 1-3x+9x^2-27x^3+\cdots+(-3x)^n+\cdots \quad \text{or} \quad \sum_{n=0}^{\infty}(-3x)^n$$

which is a geometric series where $a_1 = 1$ and $r=-3x$ So $\boxed{f'(x)=\dfrac{1}{1+3x}}$

(c)

$$\boxed{e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots}$$

$$g(x) = e^x f(x) = \left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)\left(x-\frac{3}{2}x^2+3x^3-\cdots\right)$$

$$=1\left(x-\frac{3}{2}x^2+3x^3-\cdots\right)+x\left(x-\frac{3}{2}x^2+3x^3-\cdots\right)+\frac{x^2}{2!}\left(x-\frac{3}{2}x^2+3x^3-\cdots\right)+\cdots$$

$$=x-\frac{3}{2}x^2+3x^3-\cdots+x^2-\frac{3}{2}x^3+\cdots+\frac{1}{2!}x^3+\cdots$$

$$\approx \boxed{x+\left(-\frac{3}{2}+1\right)x^2+\left(3-\frac{3}{2}+\frac{1}{2!}\right)x^3} = x-\frac{1}{2}x^2+2x^3$$

NOTE: ***The following work will not earn full credit because it did not use the Maclaurin series for e^x to find the third-degree Taylor polynomial for $e^x f(x)$ - but Mr. Calculus enjoyed working the problem this way so he left it in!***

$g(x)=e^x f(x)$	$g(0)=f(0)=0$
$g'(x)=e^x f'(x)+e^x f(x)$	$g'(0)=f'(0)+f(0)=1$
$g''(x)=e^x f''(x)+2e^x f'(x)+e^x f(x)$	$g''(0)=f''(0)+2f'(0)+f(0)=-1$

9) Ratio Test

$$\lim_{n\to\infty}\left|\frac{(-3)^{n+1-1}\,x^{n+1}}{n+1}\cdot\frac{n}{(-3)^{n-1}x^{n}}\right|$$

$$=\lim_{n\to\infty}\left|\frac{(-3)^{n}\,x^{n+1}}{n+1}\cdot\frac{n}{(-3)^{n-1}\,x^{n}}\right|$$

$$=\lim_{n\to\infty}\left|\frac{\cancel{(-3)^{n}}\cdot\cancel{x^{n}}\cdot x^{1}}{n+1}\cdot\frac{n}{\cancel{(-3)^{n}}\cdot(-3)^{-1}\cdot\cancel{x^{n}}}\right|$$

$$=\lim_{n\to\infty}\left|\frac{n}{n+1}\cdot(-3)x\right|$$

$$1\,|-3x|<1$$

$$|3x|<1$$

$$\frac{-1}{3}<\frac{3x}{3}<\frac{1}{3}$$

$$-\frac{1}{3}<x<\frac{1}{3}$$

$$\boxed{\text{radius}=\frac{1}{3}}$$

b) $f(x) = x - \frac{3}{2}x^2 + 3x^3 + \frac{(-3)^{4-1}x^4}{4}$

$$f(x) = x - \frac{3}{2}x^2 + 3x^3 - \frac{27x^4}{4} + \ldots$$

$$f'(x) = 1 - \frac{3}{\cancel{2}} \cdot \cancel{2}x + 9x^2 - 27 \cdot \frac{\cancel{4}x^3}{\cancel{4}} + \ldots$$

$$= \boxed{1 - 3x + 9x^2 - 27x^3 + \ldots (-3x)^n}$$

This is a geometric series we know that $S = \frac{a_1}{1-r}$ $\quad a_1 = 1$, $r = -3x$

$$S = \frac{1}{1++3x}$$

$$S = \boxed{\frac{1}{1+3x}}$$

c) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

$$e^x \cdot f(x) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)\left(x - \frac{3}{2}x^2 + 3x^3 \dots\right)$$

$$1\left(x - \frac{3}{2}x^2 + 3x^3 + \dots\right) + x\left(x - \frac{3}{2}x^2 + 3x^3 + \dots\right) + \frac{x^2}{2}\left(x - \frac{3}{2}x^2 + 3x^3 + \dots\right)$$

$$x \boxed{- \frac{3}{2}x^2} + 3x^3 + \dots + \boxed{1x^2} - \frac{3}{2}x^3 + 3x^4 + \dots \frac{x^3}{2} - \frac{3x^4}{4} + \frac{3x^5}{2}$$

$$x - \frac{1}{2}x^2 + \underbrace{\left(3 - \frac{3}{2} + \frac{1}{2}\right)}_{2}x^3$$

$$\boxed{x - \frac{1}{2}x^2 + 2x^3}$$