6. The Maclaurin series for e^x is $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!} + \cdots$. The continuous function f is defined by $f(x) = \frac{e^{(x-1)^2} - 1}{(x-1)^2}$ for $x \neq 1$ and $f(1) = 1$. The function f has derivatives of all orders at $x = 1$.

(a) Write the first four nonzero terms and the general term of the Taylor series for $e^{(x-1)^2}$ about $x = 1$.

(b) Use the Taylor series found in part (a) to write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

(c) Use the ratio test to find the interval of convergence for the Taylor series found in part (b).

(d) Use the Taylor series for f about $x = 1$ to determine whether the graph of f has any points of inflection.

a) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$

$e^{(x-1)^2} = 1 + (x-1)^2 + \frac{((x-1)^2)^2}{2!} + \frac{((x-1)^2)^3}{3!} + \dots \frac{((x-1)^2)^n}{n!}$

$$\boxed{e^{(x-1)^2} = 1 + (x-1)^2 + \frac{(x-1)^4}{2!} + \frac{(x-1)^6}{3!} + \dots + \frac{(x-1)^{2n}}{n!}}$$

b) $\frac{\cancel{1} + (x-1)^2}{(x-1)^2} + \frac{\frac{(x-1)^4}{2!}}{(x-1)^2} + \frac{\frac{(x-1)^6}{3!}}{(x-1)^2} + \dots + \frac{\frac{(x-1)^{2n}}{n!}}{(x-1)^2} \cancel{-1}$

$$\boxed{f(x) = 1 + \frac{(x-1)^2}{2!} + \frac{(x-1)^4}{3!} + \dots + \frac{(x-1)^{2n}}{(n+1)!}}$$

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Question 6

The Maclaurin series for e^x is $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!} + \cdots$. The continuous function f is defined by $f(x) = \frac{e^{(x-1)^2} - 1}{(x-1)^2}$ for $x \neq 1$ and $f(1) = 1$. The function f has derivatives of all orders at $x = 1$.

(a) Write the first four nonzero terms and the general term of the Taylor series for $e^{(x-1)^2}$ about $x = 1$.

(b) Use the Taylor series found in part (a) to write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

(c) Use the ratio test to find the interval of convergence for the Taylor series found in part (b).

(d) Use the Taylor series for f about $x = 1$ to determine whether the graph of f has any points of inflection.

Solution	Scoring
(a) $1 + (x-1)^2 + \frac{(x-1)^4}{2} + \frac{(x-1)^6}{6} + \cdots + \frac{(x-1)^{2n}}{n!} + \cdots$	$2: \begin{cases} 1: \text{first four terms} \\ 1: \text{general term} \end{cases}$
(b) $1 + \frac{(x-1)^2}{2} + \frac{(x-1)^4}{6} + \frac{(x-1)^6}{24} + \cdots + \frac{(x-1)^{2n}}{(n+1)!} + \cdots$	$2: \begin{cases} 1: \text{first four terms} \\ 1: \text{general term} \end{cases}$
(c) $\lim\limits_{n\to\infty} \left\lvert \dfrac{\frac{(x-1)^{2n+2}}{(n+2)!}}{\frac{(x-1)^{2n}}{(n+1)!}} \right\rvert = \lim\limits_{n\to\infty} \frac{(n+1)!}{(n+2)!}(x-1)^2 = \lim\limits_{n\to\infty} \frac{(x-1)^2}{n+2} = 0$ Therefore, the interval of convergence is $(-\infty, \infty)$.	$3: \begin{cases} 1: \text{sets up ratio} \\ 1: \text{computes limit of ratio} \\ 1: \text{answer} \end{cases}$
(d) $f''(x) = 1 + \frac{4 \cdot 3}{6}(x-1)^2 + \frac{6 \cdot 5}{24}(x-1)^4 + \cdots + \frac{2n(2n-1)}{(n+1)!}(x-1)^{2n-2} + \cdots$ Since every term of this series is nonnegative, $f''(x) \geq 0$ for all x. Therefore, the graph of f has no points of inflection.	$2: \begin{cases} 1: f''(x) \\ 1: \text{answer} \end{cases}$

c) $$\lim_{n\to\infty}\left|\frac{(x-1)^{\overset{2n+2}{2(n+1)}}}{(n+1+1)!}\cdot\frac{(n+1)!}{(x-1)^{2n}}\right| = \lim_{n\to\infty}\left|\frac{\cancel{(x-1)^{2n}}\,(x-1)^2}{(n+2)\cancel{(n+1)!}}\cdot\frac{\cancel{(n+1)!}}{\cancel{(x-1)^{2n}}}\right|$$

$$=\lim_{n\to\infty}\left|\frac{1}{n+2}\cdot(x-1)^2\right|$$

$$= \frac{0}{0}\left|(x-1)^2\right| < \frac{1}{0}$$

$$= \sqrt{\left|(x-1)^2\right|} < \sqrt{\infty}$$

$$|x-1| < \infty$$

$$-\infty < x-1 < \infty$$
$$+1 \quad +1 \quad +1$$

$$\boxed{-\infty < x < \infty}$$

d) Inflection points - look at 2nd derivative

$$f(x) = 1 + \frac{(x-1)^2}{2} + \frac{(x-1)^4}{6} + \frac{(x-1)^6}{24} + \dots + \frac{(x-1)^{2n}}{(n+1)!}$$

$$f'(x) = \frac{\cancel{2}(x-1)}{\cancel{2}} + \frac{4(x-1)^3}{6} + \frac{6(x-1)^5}{24} + \dots + \frac{2n(x-1)^{2n-1}}{(n+1)!}$$

$$f'(x) = (x-1) + \frac{2(x-1)^3}{3} + \frac{(x-1)^5}{4} + \dots + \frac{2n(x-1)^{2n-1}}{(n+1)!}$$

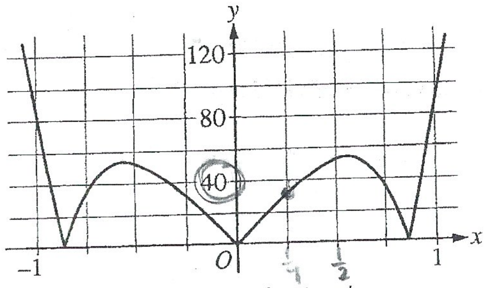
$$f''(x) = 1 + \frac{6(x-1)^2}{3} + \frac{5(x-1)^4}{4} + \dots + \frac{2n(2n-1)\,x^{2n-2}}{(n+1)!}$$

Since all the terms are positive (nonnegative)

thus $f''(x) \geq 0$ for all x

Thus it is all concave up,

$\boxed{\text{Therefore, the graph has no inflection points}}$



Graph of $y = \left|f^{(5)}(x)\right|$

6. Let $f(x) = \sin\left(x^2\right) + \cos x$. The graph of $y = \left|f^{(5)}(x)\right|$ is shown above.

(a) Write the first four nonzero terms of the Taylor series for $\sin x$ about $x = 0$, and write the first four nonzero terms of the Taylor series for $\sin\left(x^2\right)$ about $x = 0$.

(b) Write the first four nonzero terms of the Taylor series for $\cos x$ about $x = 0$. Use this series and the series for $\sin\left(x^2\right)$, found in part (a), to write the first four nonzero terms of the Taylor series for f about $x = 0$.

(c) Find the value of $f^{(6)}(0)$.

(d) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Using information from the graph of $y = \left|f^{(5)}(x)\right|$ shown above, show that $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| < \frac{1}{3000}$.

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a) $\boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots}$

$\sin(x^2) = (x^2) - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \dots = \boxed{x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots}$

b) $\boxed{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots}$

$-\frac{x^6}{6!} - \frac{120x^6}{6!}$

$f(x) = \sin(x^2) + \cos x = 1 + \frac{2x^2}{2 \cdot 1} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} - \frac{x^6 \cdot 4 \cdot 5 \cdot 6}{3! \cdot 4 \cdot 5 \cdot 6} + \dots$

$\boxed{f(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{121x^6}{6!} + \dots}$

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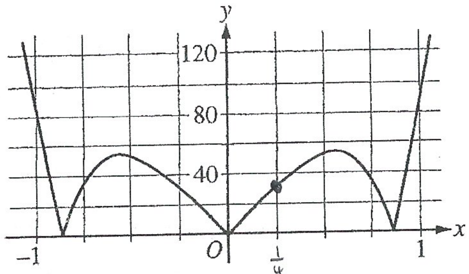
Question 6



Graph of $y = \left|f^{(5)}(x)\right|$

Let $f(x) = \sin\left(x^2\right) + \cos x$. The graph of $y = \left|f^{(5)}(x)\right|$ is shown above.

(a) Write the first four nonzero terms of the Taylor series for $\sin x$ about $x = 0$, and write the first four nonzero terms of the Taylor series for $\sin\left(x^2\right)$ about $x = 0$.

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(c) Find the value of $f^{(6)}(0)$.

(d) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Using information from the graph of $y = \left|f^{(5)}(x)\right|$ shown above, show that $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| < \frac{1}{3000}$.

(a) $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$

$\sin\left(x^2\right) = x^2 - \dfrac{x^6}{3!} + \dfrac{x^{10}}{5!} - \dfrac{x^{14}}{7!} + \cdots$

$3 : \begin{cases} 1 : \text{series for } \sin x \\ 2 : \text{series for } \sin\left(x^2\right) \end{cases}$

(b) $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots$

$f(x) = 1 + \dfrac{x^2}{2} + \dfrac{x^4}{4!} - \dfrac{121x^6}{6!} + \cdots$

$3 : \begin{cases} 1 : \text{series for } \cos x \\ 2 : \text{series for } f(x) \end{cases}$

(c) $\dfrac{f^{(6)}(0)}{6!}$ is the coefficient of x^6 in the Taylor series for f about $x = 0$. Therefore $f^{(6)}(0) = -121$.

1 : answer

(d) The graph of $y = \left|f^{(5)}(x)\right|$ indicates that $\max\limits_{0 \le x \le \frac{1}{4}} \left|f^{(5)}(x)\right| < 40$.

Therefore

$$\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| \le \frac{\max\limits_{0 \le x \le \frac{1}{4}} \left|f^{(5)}(x)\right|}{5!} \cdot \left(\frac{1}{4}\right)^5 < \frac{40}{120 \cdot 4^5} = \frac{1}{3072} < \frac{1}{3000}.$$

$2 : \begin{cases} 1 : \text{form of the error bound} \\ 1 : \text{analysis} \end{cases}$

c) Find $f^{6}(0)$ term looks like $\frac{f^{6}(0)x^{6}}{6!}$

we know $\frac{(-121)x^{6}}{6!}$

Thus $f^{6}(0) = \boxed{-121}$

d)

d) error $R_n(x) = \frac{f^{n+1}(z)(x-c)^{n+1}}{(n+1)!}$

$n=4$

$c=0$

$x=\frac{1}{4}$

$$R_4(x) = \frac{f^5(z)(x-0)^5}{5!}$$

$$R_4(x) = \frac{f^5(z)x^5}{5!}$$

$$R_4\left(\tfrac{1}{4}\right) = \frac{f^5(z)\left(\tfrac{1}{4}\right)^5}{5!}$$

look at graph

$f^5\left(\tfrac{1}{4}\right) < 40$

$$R_4\left(\tfrac{1}{4}\right) \le \frac{40\left(\tfrac{1}{4}\right)^5}{5!} = 0.0003260$$

$$\boxed{R_4\left(\tfrac{1}{4}\right) \le 0.0003260 < 0.000333}$$

error

$$\left|P_4\left(\tfrac{1}{4}\right) - f\left(\tfrac{1}{4}\right)\right| < $$

real - approx 4th term

5th term

$$\frac{f^5\left(\tfrac{1}{4}\right)x^5}{5!}$$

$$\frac{40\cdot\left(\tfrac{1}{4}\right)^5}{5!}$$

$$\frac{40}{4^5\cdot 5!} = \frac{1}{3072} < \frac{1}{3000}$$

SERIES

2011 AP® CALCULUS BC FREE-RESPONSE QUESTIONS (Form B)

6. Let $f(x) = \ln\left(1 + x^3\right)$.

(a) The Maclaurin series for $\ln(1 + x)$ is $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1} \cdot \frac{x^n}{n} + \cdots$. Use the series to write the first four nonzero terms and the general term of the Maclaurin series for f.

(b) The radius of convergence of the Maclaurin series for f is 1. Determine the interval of convergence. Show the work that leads to your answer.

(c) Write the first four nonzero terms of the Maclaurin series for $f'\left(t^2\right)$. If $g(x) = \int_0^x f'\left(t^2\right)\,dt$, use the first two nonzero terms of the Maclaurin series for g to approximate $g(1)$.

(d) The Maclaurin series for g, evaluated at $x = 1$, is a convergent alternating series with individual terms that decrease in absolute value to 0. Show that your approximation in part (c) must differ from $g(1)$ by less than $\frac{1}{5}$.

AP® CALCULUS BC
2011 SCORING GUIDELINES (Form B)

Question 6

Let $f(x) = \ln(1 + x^3)$.

(a) The Maclaurin series for $\ln(1+x)$ is $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1} \cdot \frac{x^n}{n} + \cdots$. Use the series to write the first four nonzero terms and the general term of the Maclaurin series for f.

(b) The radius of convergence of the Maclaurin series for f is 1. Determine the interval of convergence. Show the work that leads to your answer.

(c) Write the first four nonzero terms of the Maclaurin series for $f'(t^2)$. If $g(x) = \int_0^x f'(t^2)\,dt$, use the first two nonzero terms of the Maclaurin series for g to approximate $g(1)$.

(d) The Maclaurin series for g, evaluated at $x = 1$, is a convergent alternating series with individual terms that decrease in absolute value to 0. Show that your approximation in part (c) must differ from $g(1)$ by less than $\frac{1}{5}$.

Solution	Scoring
(a) $x^3 - \frac{x^6}{2} + \frac{x^9}{3} - \frac{x^{12}}{4} + \cdots + (-1)^{n+1} \cdot \frac{x^{3n}}{n} + \cdots$	$2 : \begin{cases} 1 : \text{first four terms} \\ 1 : \text{general term} \end{cases}$
(b) The interval of convergence is centered at $x = 0$. At $x = -1$, the series is $-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots - \frac{1}{n} - \cdots$, which diverges because the harmonic series diverges. At $x = 1$, the series is $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n+1} \cdot \frac{1}{n} + \cdots$, the alternating harmonic series, which converges. Therefore the interval of convergence is $-1 < x \le 1$.	2 : answer with analysis
(c) The Maclaurin series for $f'(x)$, $f'(t^2)$, and $g(x)$ are $f'(x) : \sum_{n=1}^{\infty} (-1)^{n+1} \cdot 3x^{3n-1} = 3x^2 - 3x^5 + 3x^8 - 3x^{11} + \cdots$ $f'(t^2) : \sum_{n=1}^{\infty} (-1)^{n+1} \cdot 3t^{6n-2} = 3t^4 - 3t^{10} + 3t^{16} - 3t^{22} + \cdots$ $g(x) : \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{3x^{6n-1}}{6n-1} = \frac{3x^5}{5} - \frac{3x^{11}}{11} + \frac{3x^{17}}{17} - \frac{3x^{23}}{23} + \cdots$ Thus $g(1) \approx \frac{3}{5} - \frac{3}{11} = \frac{18}{55}$.	$4 : \begin{cases} 1 : \text{two terms for } f'(t^2) \\ 1 : \text{other terms for } f'(t^2) \\ 1 : \text{first two terms for } g(x) \\ 1 : \text{approximation} \end{cases}$
(d) The Maclaurin series for g evaluated at $x = 1$ is alternating, and the terms decrease in absolute value to 0. Thus $\left\lvert g(1) - \frac{18}{55} \right\rvert < \frac{3 \cdot 1^{17}}{17} = \frac{3}{17} < \frac{1}{5}$.	1 : analysis

d) $R_n(x) = \dfrac{f^{n+1}(z)(x-c)^{n+1}}{(n+1)!}$

$n=2$

$c=0$

$x=1$

$z=$

$0<z<1$

$R_2(x) = \dfrac{f^3(z)\,x^3}{3!}$ 3rd term

$\dfrac{3x^{6\cdot3-1}}{(6\cdot3-1)} = \dfrac{3x^{17}}{17}$

$R_2(x) = \dfrac{3x^{17}}{17} = \dfrac{3(1)^{17}}{17} = 0.17647 < \dfrac{1}{5} = .2$

$\left| g(1) - \underbrace{\dfrac{190}{55}}_{\text{2nd term approx}} \right| < \dfrac{3x^{17}}{17}$ 3rd term

$\dfrac{3(1)^{17}}{17}$

$\left\| \text{error using 2nd term} \right\| <$ next term $= \dfrac{3}{17} < \dfrac{1}{5}$

6. $f(x) = \ln(1+x^3)$

a) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + (-1)^{n+1} \cdot \frac{x^n}{n}$

$$\ln(1+x^3) = (x^3) - \frac{(x^3)^2}{2} + \frac{(x^3)^3}{3} - \frac{(x^3)^4}{4} + \dots + (-1)^{n+1} \frac{(x^3)^n}{n}$$

$$\boxed{\ln(1+x^3) = x^3 - \frac{x^6}{2} + \frac{x^9}{3} - \frac{x^{12}}{4} + \dots + (-1)^{n+1} \frac{x^{3n}}{n}}$$

b) ratio Test

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1+1} x^{3(n+1)}}{n+1} \cdot \frac{n}{(-1)^{n+1} x^{3n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)^{n+2} x^{3n+3}}{n+1} \cdot \frac{n}{(-1)^{n+1} x^{3n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)^{n+2} \cdot \cancel{x^{3n}} \cdot x^3}{n+1} \cdot \frac{n}{\cancel{(-1)^{n+1}} \cancel{x^{3n}}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n}{n+1} \cdot x^3 \right|$$

$$1 \cdot |x^3| < 1$$

$$|\sqrt[3]{x^3}| < \sqrt[3]{1}$$

$|x| < 1$

Thus, $-1 < x \leq 1$

*check endpoints

$x = 1$ $\quad (-1)^{n+1} \frac{1^{3n}}{n} = (-1)^{n+1} \frac{1}{n}$

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$

alt. series

① smaller terms

② $\lim_{n\to\infty} = 0$

converge

$x = -1$ $\quad \frac{(-1)^{n+1}(-1)^{3n}}{n}$

$\frac{(-1)^{4n+1}}{n}$

$-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4}$

$-1\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right)$

p-series

Diverges

c) $f(x) = x^3 - \frac{x^6}{2} + \frac{x^9}{3} - \frac{x^{12}}{4} + \dots + (-1)^{n+1} \cdot \frac{x^{3n}}{n}$

$f'(x) = 3x^2 - \frac{6x^5}{2} + \frac{9x^8}{3} - \frac{12x^{11}}{4} + \dots + (-1)^{n+1} \cdot \frac{\cancel{3n}\, x^{3n}}{\cancel{n}}$

$f'(x) = 3x^2 - 3x^5 + 3x^8 - 3x^{11} + \dots + (-1)^{n+1}\, 3x^{3n-1}$

$f'(t^2) = 3(t^2)^2 - 3(t^2)^5 + 3(t^2)^8 - 3(t^2)^{11} + \dots + (-1)^{n+1}\, 3 \cdot (t^2)^{3n-1}$

$f'(t^2) = 3t^4 - 3t^{10} + 3t^{16} - 3t^{22} + \dots + (-1)^{n+1}\, 3t^{6n-2}$

$g(x) = \int_0^x f'(t^2)\,dt$ Integrate each term and plug in x

$g(x) = \frac{3x^5}{5} - \frac{3x^{11}}{11} + \frac{3x^{17}}{17} - \frac{3x^{23}}{23} + \dots + (-1)^{n+1} \frac{3x^{6n-1}}{6n-1}$

$g(1) = \frac{3(1)^5}{5} - \frac{3(1)^{11}}{11} = \frac{3 \cdot 11}{5 \cdot 11} - \frac{3 \cdot 5}{11 \cdot 5} = \frac{33}{55} - \frac{15}{55} = \boxed{\frac{18}{55}}$