DIFFERENTIAL EQUATIONS, SLOPEFIELDS, EULER'S METHOD

2013 AP® CALCULUS BC FREE-RESPONSE QUESTIONS

5. Consider the differential equation $\frac{dy}{dx} = y^2(2x + 2)$. Let $y = f(x)$ be the particular solution to the differential equation with initial condition $f(0) = -1$.

(a) Find $\lim_{x \to 0} \frac{f(x) + 1}{\sin x}$. Show the work that leads to your answer.

(b) Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f\left(\frac{1}{2}\right)$.

(c) Find $y = f(x)$, the particular solution to the differential equation with initial condition $f(0) = -1$.

$0 \xrightarrow{\frac{1}{4}+\frac{1}{4}} \frac{1}{2}$

a) $\lim_{x \to 0} \frac{f(x)+1}{\sin x}$ f is differential at $x=0$ (given) and f is continuous at $x=0$ since $f(0)=-1$

plug in

$f(0)=-1$

$\lim_{x\to 0} \frac{f(x)+1}{\sin x} = \frac{f(0)+1}{\sin(0)} = \frac{-1+1}{0} = \frac{0}{0}$

Thus, we can apply L'Hopitals Rule

$\lim_{x\to 0} \frac{f'(x)}{\cos x} = \frac{f'(0)}{\cos(0)} = \frac{2}{1} = \boxed{2}$

$(0,-1)$

$\frac{dy}{dx} = (-1)^2(2(0)+2)$
$= 1(2) = 2$

$y^2(2x+2)$

b)

(x,y)	$\Delta x/dx$	$\frac{dy}{dx}$	$dx\left(\frac{dy}{dx}\right)=dy$	$(x+dx,\ y+dy)$
$(0,-1)$	$\frac{1}{4}$	2	$.25(2) = \frac{1}{2}$	$0+\frac{1}{4}$, $-1+\frac{1}{2}$ $\left(\frac{1}{4}, -\frac{1}{2}\right)$
$\left(\frac{1}{4}, -\frac{1}{2}\right)$	$\frac{1}{4}$	$\frac{5}{8}$	$\frac{1}{4}\cdot\frac{5}{8}$ $\frac{5}{32}$	$\frac{1}{4}+\frac{1}{4}$, $-\frac{1}{2}+\frac{5}{32}$ $\left(\frac{1}{2}, \frac{-11}{32}\right)$

$f\left(\frac{1}{2}\right) = \boxed{\frac{-11}{32}}$

$1^2(2(0)+2)$
$\frac{1}{4}\cdot 2 = \frac{1}{2}$

$\left(-\frac{1}{2}\right)^2\left(2\left(\frac{1}{4}\right)+2\right)$
$\frac{1}{4}\left(\frac{1}{2}+2\right)$
$\frac{1}{4}\cdot\frac{5}{2} = \frac{5}{8}$
$\frac{-16}{32}\ \frac{+5}{32}$

c) $\frac{dy}{dx} = y^2(2x+2)$ $f(0)=-1$

$\int \frac{dy}{y^2} = \int (2x+2)\,dx$

$-\frac{1}{y} = \frac{\cancel{2}x^2}{\cancel{2}} + 2x + C$

$\frac{-1}{-1} = 0^2 + 2(0) + C$ $C=1$

$-\frac{1}{y} = x^2 + 2x + 1$

$\frac{1}{y} = -x^2 - 2x - 1$

$y = \frac{1}{-x^2-2x-1} = \boxed{y = \frac{-1}{x^2+2x+1}}$

5. Consider the differential equation $\frac{dy}{dx} = 1 - y$. Let $y = f(x)$ be the particular solution to this differential equation with the initial condition $f(1) = 0$. For this particular solution, $f(x) < 1$ for all values of x.

(a) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(0)$. Show the work that leads to your answer.

(b) Find $\lim_{x \to 1} \frac{f(x)}{x^3 - 1}$. Show the work that leads to your answer.

(c) Find the particular solution $y = f(x)$ to the differential equation $\frac{dy}{dx} = 1 - y$ with the initial condition $f(1) = 0$.

a) $\frac{dy}{dx} = 1-y$

(x,y)	$\Delta x / dx$	$\frac{dy}{dx}$	$dx\left(\frac{dy}{dx}\right) = dy$	$(x+dx, y+dy)$
$(1,0)$	$-.5$	$1-0$ 1	$-.5(1)$ $-.5$	$1+-.5 \quad 0-.5$ $(.5, -.5)$
$(.5,-.5)$	$-.5$	$1+.5$ 1.5	$-.5(1.5)$ $-.75$	$.5-.5, -.5-.75$ $(0, -1.25)$

$f(0) = \boxed{-1.25}$

b) $\lim_{x \to 1} \frac{f(x)}{x^3-1}$ f is differential at $x=1$ (given) and f is continuous at $x=1$ since $f(1)=0$.

Thus, if we plug in $\lim_{x \to 1} \frac{f(x)}{x^3-1} = \frac{f(1)}{1^3-1} = \boxed{\frac{0}{0}}$ $f(1)=0$.

Thus, we can apply L'Hopitals Rule

$\lim_{x \to 1} \frac{f'(x)}{3x^2} = \frac{f'(1)}{3(1)^2} = \boxed{\frac{1}{3}}$

slope at $x=1$ is $\frac{dy}{dx} = 1$ from table

$\frac{dy}{dx} = 1-0 = 1$ from formula

c) $\frac{dy}{dx} = 1-y$

$\int \frac{dy}{1-y} = \int dx$

$-\ln|1-y| = x+C$

$f(1)=0$

Find C

$-\ln|1-0| = 1+C$

$-\ln 1 = 1+C$

$0 = 1+C$

$C=-1$

$-\ln|1-y| = x-1$

$e^{\ln|1-y|} = e^{-x+1}$

$1-y = e^{1-x}$ -1

$-y = e^{1-x} - 1$

$\boxed{y = 1-e^{1-x}}$