BC #1

38

38. You wish to estimate e^x, over the interval $|x| < 2$, with an error less than 0.001. The Lagrange error term suggests that you use a Taylor polynomial at 0 with degree at least

(A) 6 (B) 9 (C) 10 (D) 11 (E) 12

we know that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \quad + \frac{x^n}{n!} + \dots$$

The Lagrange remainder R, after n terms, for some c in the interval $0<c<x$ definition is

$$R = \frac{f^{(n+1)}(c) \cdot c^{n+1}}{(n+1)!} \quad \text{for } 0<c<x$$

Now lets plug in our information

$f(x)=e^x$, so $f(c) = e^c$, & since $|x|<2$

$$R = \frac{e^c \cdot c^{n+1}}{(n+1)!} \qquad 0<c<2$$

so, if $0<c<2$, then R would be the greatest value when $c=2$, so plug in $c=2$ and n needs to satisfy

given error

$$\frac{e^2 \cdot 2^{n+1}}{(n+1)!} < 0.001$$

now plug in different values for n.

~~A)~~ $n=6$ $\frac{e^2 \cdot 2^{6+1}}{(6+1)!} = .187$ not < 0.001

~~B)~~ $n=9$ $\frac{e^2 \cdot 2^{9+1}}{(9+1)!} = .0021$ not < 0.001

c) $n=10$ $\frac{e^2 \cdot 2^{10+1}}{(10+1)!} =$ 0.000379

which is less that 0.001

$\boxed{C}$

BC #1

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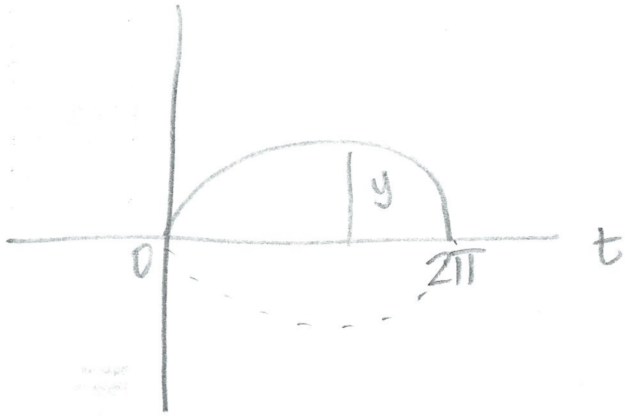
39. Find the volume of the solid formed when one arch of the cycloid defined parametrically by $x = \theta - \sin\theta$, $y = 1 - \cos\theta$ is rotated around the x-axis.

(A) 15.708 (B) 17.306 (C) 19.739 (D) 29.609 (E) 49.348

change MODE : Func. to PAR

graph $x_1 = T - \sin T$

$y_1 = 1 - \cos T$



Find area of rotation is a circle

πr^2 where $r = y$

$$\int_0^{2\pi} \pi y^2\,dx \qquad y = 1-\cos t$$

Find derivative of x

$x = t - \sin t$

$dx = (1 - \cos t)\,dt$

$$\pi \int_0^{2\pi} (1-\cos t)^2 (1-\cos t)\,dt$$

$$\pi \int_0^{2\pi} (1-\cos t)^3\,dt$$

plug into calculator

$\boxed{49.348}$ E

BC #1

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40. Which definite integral represents the length of the first quadrant arc of the curve defined by $x(t) = e^t$, $y(t) = 1 - t^2$?

(A) $\int_{-1}^{1} \sqrt{1 + \frac{4t^2}{e^{2t}}}\, dt$ (B) $\int_{1/e}^{e} \sqrt{1 + \frac{4t^2}{e^{2t}}}\, dt$ (C) $\int_{-1}^{1} \sqrt{e^{2t} + 4t^2}\, dt$

(D) $\int_{0}^{1} \sqrt{e^{2t} + 4t^2}\, dt$ (E) $\int_{1/e}^{e} \sqrt{e^{2t} + 4t^2}\, dt$

For the first quadrant, we know that both x and y must be positive

* we know that $x = e^t$ is positive for all t
* we know that $y = 1 - t^2$ is only positive when $-1 < t < 1$

The formula for arc length is

$$\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$x = e^t$ $y = 1 - t^2$

$\boxed{\frac{dx}{dt} = e^t}$ $\boxed{\frac{dy}{dt} = -2t}$ plug into formula.

when both 1 are positive

$$\int_{-1}^{1} \sqrt{(e^t)^2 + (-2t)^2}\, dt$$

$$\boxed{\int_{-1}^{1} \sqrt{e^{2t} + 4t^2}\, dt}$$ C

BC #1

41

41. For which function is $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ the Taylor series about 0?

(A) e^x (B) e^{-x} (C) $\sin x$ (D) $\cos x$ (E) $\ln(1+x)$

plug in 0

$$\frac{(-1)^0 x^{2(0)}}{(2(0))!} = 1$$

plug in 1

$$\frac{(-1)^1 x^{2(1)}}{(2(1))!} = \frac{-1x^2}{2!}$$

plug in 2

$$\frac{(-1)^2 x^{2\cdot 2}}{(2\cdot 2)!} = \frac{x^4}{4!}$$

plug in 3

$$\frac{(-1)^3 x^{2\cdot 3}}{(2\cdot 3)!} = \frac{-x^6}{6!}$$

what function is

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

I know that cos and sin both skip terms so lets check cosx first

$f(0) = \cos(0) = 1$

$\frac{f'(0)x}{1!} = -\sin(0) = \frac{0x}{1!} = 0$

$\frac{f''(0)x^2}{2!} = -\cos(0) = \frac{-1x^2}{2!}$

Thus this matchs above

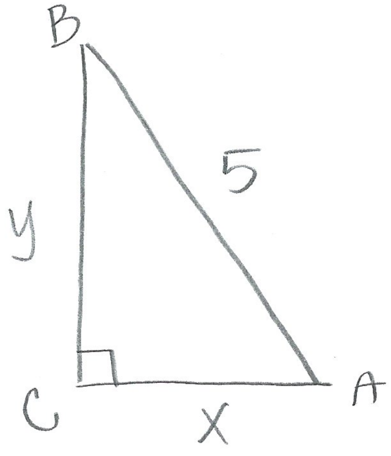
$\boxed{\cos x}$ D

BC #1

42

42. The hypotenuse AB of a right triangle ABC is 5 feet, and one leg, AC, is decreasing at the rate of 2 feet per second. The rate, in square feet per second, at which the area is changing when $AC = 3$ is

(A) $\frac{25}{4}$ (B) $\frac{7}{4}$ (C) $-\frac{3}{2}$ (D) $-\frac{7}{4}$ (E) $-\frac{7}{2}$



we know that $AB = 5$

$AC = X$ and $BC = y$

since AC is decreasing at a rate of 2 ft per sec. ($AC = x$)

$\frac{dx}{dt} = -2$

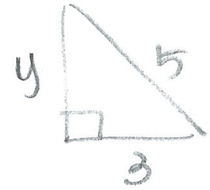
we want to know the rate the area is changing so area of a triangle is $\frac{1}{2}bh$

$A = \frac{1}{2}x \cdot y$, we want to know $\frac{dA}{dt}$

so find the derivative of $A = \frac{1}{2}x \cdot y$ with respect to time.

product rule $fg' + gf'$

$\frac{dA}{dt} = \frac{1}{2}\left[x \cdot 1\frac{dy}{dt} + y \cdot 1\frac{dx}{dt}\right]$



$AC = 3$

now plug in $\frac{dx}{dt} = -2$, $x = 3$, $y = 4$

$y^2 + 3^2 = 5^2$

$y^2 + \cancel{9} = 25$

$-\cancel{9} \quad -9$

$\sqrt{y^2} = \sqrt{16}$

$y = 4$

*we need to find $\frac{dy}{dt}$, lets write an equation for y

Pythagorean thm $y^2 + x^2 = 5^2 \Rightarrow y^2 = 25 - x^2$

$y = \sqrt{25 - x^2}$

$\frac{dy}{dt} = \frac{1}{2}(25 - x^2)^{-1/2} \cdot -2x$

plug in $x = 3$ $\frac{dy}{dt} = \frac{3}{2}$

$\frac{dA}{dt} = \frac{1}{2}\left[3 \cdot \frac{3}{2} + 4 \cdot -2\right] = \frac{9}{4} - \frac{4 \cdot 4}{1 \cdot 4}$

$= \frac{9}{4} - \frac{16}{4} = \boxed{-\frac{7}{4}}$

BC #1

43

43. At how many points on the interval $[0,\pi]$ does $f(x) = 2\sin x + \sin 4x$ satisfy the Mean Value Theorem?

(A) none (B) 1 (C) 2 (D) 3 (E) 4

graph $f(x) = 2\sin x + \sin 4x$

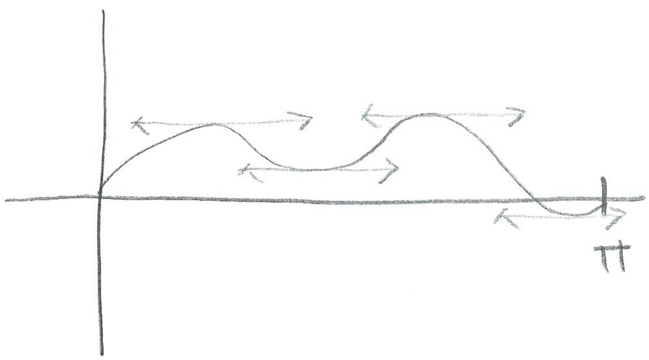
mean value theorem

given that $f(a) = f(b) = k$, then there is a number c, between a and b such that $f'(c) = 0$.

Since $f(0) = f(\pi) = 0$

and f is both continuous and differentiable

looking at the graph



we are looking for when $f' = 0$. (slope $= 0$)

we can see that there are 4 places when

$f' = 0$

thus, $\boxed{4}$ E

BC #1

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44. Which one of the following series converges?

(A) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ (B) $\sum_{n=1}^{\infty} \frac{1}{n}$ (C) $\sum_{n=1}^{\infty} \frac{1}{2n+1}$

(D) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ (E) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

*recall: p-series $\frac{1}{n^p}$ $p \le 1$ diverges, $p > 1$ converges

~~A)~~ $\frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}}$ $p = \frac{1}{2} \le 1$ Diverges

~~B)~~ $\frac{1}{n^1}$ $p = 1 \le 1$ diverges

~~C)~~ $\frac{1}{2n+1} = \frac{1}{n^1}$ $p = 1 \le 1$ Diverges

~~D)~~ $\frac{n}{n^2+1} = \frac{n}{n^2} = \frac{1}{n^1}$ $p = 1 \le 1$ Diverges

E) $\frac{1}{n^2+1} = \frac{1}{n^2}$ $p = 2 > 1$ converges

$\boxed{E}$

BC#1

45

45. The rate at which a purification process can remove contaminants from a tank of water is proportional to the amount of contaminant remaining. If 20% of the contaminant can be removed during the first minute of the process and 98% must be removed to make the water safe, approximately how long will the decontamination process take?

(A) 2 min (B) 5 min (C) 18 min (D) 20 min (E) 40 min

let Q = the amount of contaminants in the tank.

let Q_0 = the initial amount

we are going to use the formula

$$Q(t) = Q_0 e^{kt}$$

at $t=1$
20% are removed
so 80% are left

$$Q(1) = Q_0 e^{k(1)} = .80\, Q_0$$

80% of initial contaminants left

$$\cancel{Q_0} e^{k} = .80\, \cancel{Q_0}$$

$$\ln e^{k} = \ln .80$$

$$k = -.223$$

we want to find t
when 98% are moved
or 2% is left

2% of initial contaminants left

$$\cancel{Q_0} e^{-.223t} = .02\, \cancel{Q_0}$$

$$\ln e^{-.223t} = \ln .02$$

$$\frac{-.223t}{-.223} = \frac{-3.912}{-.223}$$

$$\boxed{t = 17.54}$$

C