SECTION II

Part A TIME: 30 MINUTES
2 PROBLEMS

A graphing calculator is required for some of these problems. See instructions on page 4.

1. A curve is defined by $x^2y - 3y^2 = 48$.

(a) Verify that $\dfrac{dy}{dx} = \dfrac{2xy}{6y - x^2}$.

(b) Write an equation of the line tangent to this curve at (5,3).

(c) Using your equation from part (a), estimate the y-coordinate of the point on the curve where $x = 4.93$.

(d) Show that this curve has no horizontal tangent lines.

2. The table shows the depth of water, W, in a river, as measured at 4-hour intervals during a day-long flood. Assume that W is a differentiable function of time t.

t (hr)	0	4	8	12	16	20	24
$W(t)$ (ft)	32	36	38	37	35	33	32

(a) Find the approximate value of $W'(16)$. Indicate units of measure.

(b) Estimate the average depth of the water, in feet, over the time interval $0 \le t \le 24$ hours by using a trapezoidal approximation with subintervals of length $\Delta t = 4$ days.

(c) Scientists studying the flooding believe they can model the depth of the water with the function $F(t) = 35 - 3\cos\left(\dfrac{t+3}{4}\right)$, where $F(t)$ represents the depth of the water, in feet, after t hours. Find $F'(16)$ and explain the meaning of your answer, with appropriate units, in terms of the river depth.

(d) Use the function F to find the average depth of the water, in feet, over the time interval $0 \le t \le 24$ hours.

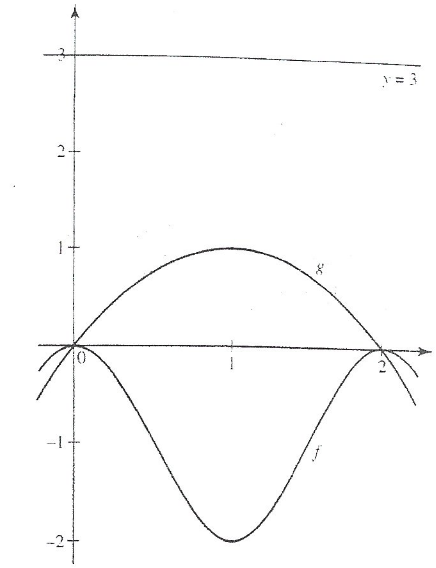
STOP

END OF PART A, SECTION II

Part B TIME: 60 MINUTES
4 PROBLEMS

No calculator is allowed for any of these problems.
If you finish Part B before time has expired, you may return to work on Part A, but you may not use a calculator.

3. The region $\boldsymbol{R}$ is bounded by the curves $f(x) = \cos(\pi x) - 1$ and $g(x) = x(2 - x)$, as shown in the figure.

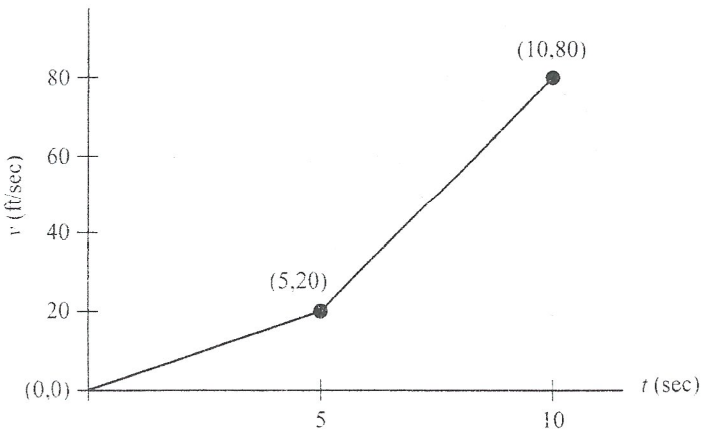


(a) Find the area of $\boldsymbol{R}$.

(b) A solid has base $\boldsymbol{R}$, and each cross section perpendicular to the x-axis is an isosceles right triangle whose hypotenuse lies in $\boldsymbol{R}$. Set up, but do not evaluate, an integral for the volume of this solid.

(b) Set up, but do not evaluate, an integral for the volume of the solid formed when $\boldsymbol{R}$ is rotated around the line $y = 3$.

4. Two autos, P and Q, start from the same point and race along a straight road for 10 seconds. The velocity of P is given by $v_p(t) = 6\left(\sqrt{1+8t} - 1\right)$ feet per second. The velocity of Q is shown in the graph.



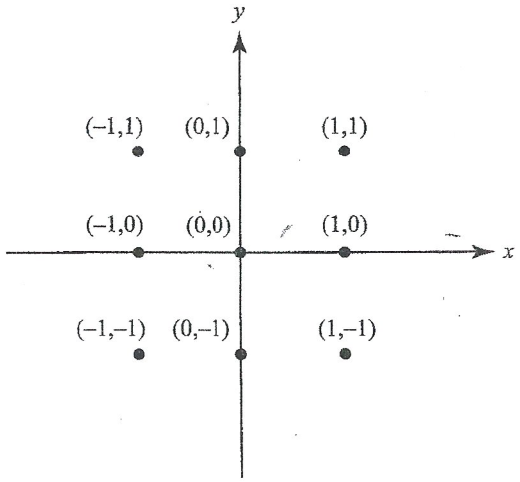
(a) At what time is P's actual acceleration (in ft/sec²) equal to its average acceleration for the entire race?

(b) What is Q's acceleration (in ft/sec²) then?

(c) At the end of the race, which auto was ahead? Explain.

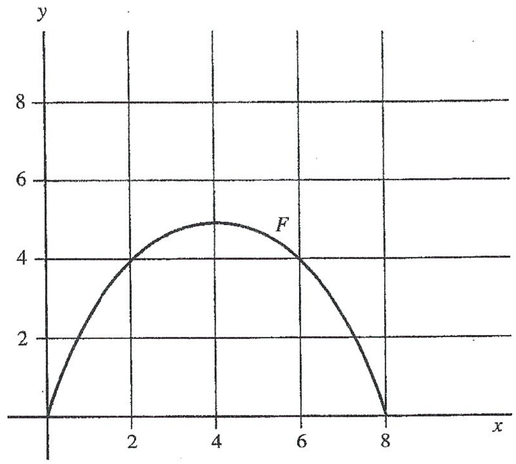
5. Given the differential equation $\dfrac{dy}{dx} = 2x\left(y^2 + 1\right)$

(a) Sketch the slope field for this differential equation at the points shown in the figure.



(b) Let f be the particular solution to the differential equation whose graph passes through $(0,1)$. Express f as a function of x, and state its domain.

6. The graph shown is for $F(x)=\int_0^x f(t)dt$.



(a) What is $\int_0^2 f(t)dt$?

(b) What is $\int_2^7 f(t)\,dt$?

(c) At what value of x does $f(x) = 0$?

(d) Over what interval is $f'(x)$ negative?

(e) Let $G(x)=\int_2^x f(t)dt$. Sketch the graph of G on the same axes.

STOP

END OF TEST

FRQ - AP Practice Exam #3

$fg' + gf'$

① $x^2y - 3y^2 = 48$

a) $x^2 \cdot 1\,dy + 2x\,dx \cdot y - 6y \cdot dy = 0$

$x^2 dy + 2xy\,dx - 6y\,dy = 0$

$\frac{dy(x^2 - 6y)}{dx} = -2xy\frac{dx}{dx}$

$\frac{dy}{dx}\frac{\cancel{(x^2-6y)}}{\cancel{x^2-6y}} = \frac{-2xy}{x^2-6y}$

$\frac{dy}{dx} = \frac{-2xy}{x^2-6y}\frac{(-1)}{(-1)}$

$$\boxed{\frac{dy}{dx} = \frac{2xy}{6y - x^2}}$$

b) $\frac{dy}{dx} = \frac{2xy}{6y-x^2}$ is the slope plug in $(5,3)$ to find the actual slope

$\frac{dy}{dx} = \frac{2(5)(3)}{6(3)-5^2} = \frac{30}{18-25} = \frac{30}{-7} = -\frac{30}{7}$

$y = mx + b$ plug in $(\overset{x}{5}, \overset{y}{3})$ and $m = 30/7$

$3 = \frac{30}{7} \cdot 5 + b$ $\quad$ $b = \frac{-129}{7}$ $\quad$ $\boxed{y = \frac{30}{7}x - \frac{129}{7}}$

$\frac{21}{7}\ \frac{3}{1 \cdot 7} = \frac{150}{7} + b$

$-\frac{150}{7}$ $\quad$ $-150/7$

or

180
-21
129

$y - y_1 = m(x - x_1)$

$$\boxed{y - 3 = \frac{30}{7}(x - 5)}$$

c) $y - 3 = \frac{30}{7}(4.93 - 5)$

$y - \cancel{3} = 0.3$
$ + \cancel{3} \quad\quad +3$

$$\boxed{y = 3.3}$$

d) Horizontal tangents happen when $\frac{dy}{dx} = 0$

$\frac{2xy}{6y - x^2} = 0$

only could happen if
$2xy = 0$, then $x = 0$ or $y = 0$
find the other coordinate $x^2y - 3y^2 = 48$

if $y = 0$, $x^2 \cdot 0 - 3 \cdot 0^2 = 48$
$0 = 48$ impossible

If $x = 0$, $0^2 \cdot y - 3y^2 = 48$
$\frac{-\cancel{3}y^2}{-\cancel{3}} = \frac{48}{-3}$
$y^2 = -16$ impossible

Thus, no horizontal tangents

(2) a) $W'(16)$ Find the slope at 16

$$\frac{y_1-y_2}{x_1-x_2}$$

choose $(12,37),(16,35)$

$$\frac{37-35}{12-16} = \frac{2}{-4} = \boxed{-\frac{1}{2}}$$

choose $(12,37)\ (20,33)$

$$\frac{37-33}{12-20} = \frac{5}{-8} = \boxed{-\frac{5}{8}}$$

choose $(16,35)\ (20,33)$

$$\frac{35-33}{16-20} = \frac{+2}{-4} = \boxed{-\frac{1}{2}}$$

b) trapezoidal rule

height equal lengths

$$T(n) = \frac{h}{2}(y_0 + 2y_1 + 2y_2 + \dots + 2y_{n-1} + y_n)$$

$h=4$

$$\frac{4}{2}(32 + 2(36) + 2(38) + 2(37) + 2(35) + 2(33) + 32)$$

$$= 2(422)$$

$$= \frac{844}{24} = \boxed{35.167 \text{ ft}}$$

↑ 24 hours

average per hour

c) $F(t) = 35 - 3\cos\left(\frac{t+3}{4}\right)$

nDeriv $\left(35 - 3\cos\left(\frac{t+3}{4}\right), x, 16)\right)$

$= \boxed{-.749}$ explain

after 16 hrs, the river depth is dropping at a rate of .749 ft/hr

d) average depth is integral

fnInt $\int_0^{24} \left(35 - 3\cos\left(\frac{t+3}{4}\right), x, 0, 24\right)$

$= \frac{842.779}{24}$ average

\# of hrs

$\boxed{35.116 \text{ ft}}$

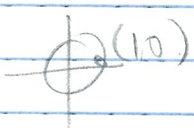
③ a) $\int_0^2 \left[x(2-x) - (\cos(\pi x) - 1)\right]$

$u = \pi x$

$\frac{du}{\pi} = \frac{\pi\,dx}{\pi}$

$\int_0^2 \left[2x - x^2 - \cos(\pi x) + 1\right]$

$\left[\frac{2x^2}{2} - \frac{x^3}{3} - \frac{1}{\pi}\sin(\pi x) + x\right]_0^2$



$\left[x^2 - \frac{x^3}{3} - \frac{1}{\pi}\sin(\pi x) + x\right]_0^2$

$\left(4 - \frac{8}{3} - \cancel{\frac{1}{\pi}\sin 2\pi} + 2\right) - \left(\cancel{0} - \cancel{\frac{0}{3}} - \cancel{\frac{1}{\pi}\sin(0\pi)} + 0\right)$

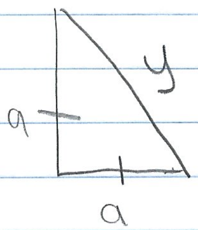
$\left(4 - \frac{8}{3} + 2\right)$

$\frac{6}{1}\cdot\frac{3}{3} - \frac{8}{3} = \frac{18}{3} - \frac{8}{3} = \boxed{\frac{10}{3}}$

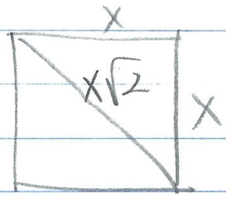
$a^2+a^2=y^2$

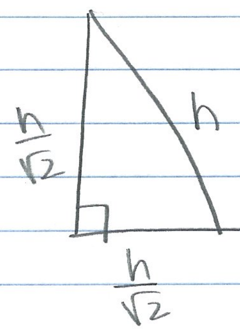
$\frac{2a^2}{2}=\frac{y^2}{2}$

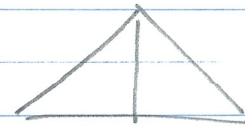
$\sqrt{a^2}=\sqrt{\frac{y^2}{2}} = \frac{y}{\sqrt{2}} =$



b)







$\frac{1}{2}bh$

$\frac{1}{2}\frac{h}{\sqrt{2}}\cdot\frac{h}{\sqrt{2}}$

$A=\frac{h^2}{4}$

of isosceles Δ given the hypotenuse

$$\int_0^2 \frac{h^2}{4}\,dx$$

$$\int_0^2 \frac{(g(x)-f(x))^2}{4}\,dx$$

$$\boxed{\frac{1}{4}\int_0^2 \left[x(2-x)-(\cos(\pi x)-1)\right]^2 dx}$$

c)

$\pi R^2-\pi r^2 = \pi(\underset{\text{top}-}{R^2}-\underset{\text{top}}{r^2})$ $\pi((3-R^2)-(3-r^2)$

$$\int_0^2 \pi(3-f(x)^2-(3-g(x)^2))$$

$$\boxed{\int_0^2 \pi\left[((3-(\cos(\pi x)-1))^2-(3-x(2-x))^2\right)dx}$$

④ a) $V_p(t) = 6(\sqrt{1+8t} - 1)$ $\qquad 6\sqrt{1+8t} - 6$

average = slope $t=0$ to $t=10$

$V_p(0) = 6(\sqrt{1+8(0)} - 1)$
$= 6(\sqrt{1} - 1) = 6(0) = 0 \qquad (0,0)$

$V_p(10) = 6(\sqrt{1+8(10)} - 1$
$= 6(\sqrt{81} - 1)$
$= 6(9-1) = 6(8) = 48 \qquad (10,48)$

slope = $\frac{48-0}{10-0} = \frac{48 \div 2}{10 \div 2} = \frac{24}{5}$

$a_p(t) = V'_p(t) = \cancel{6}^{3} \frac{1}{\cancel{2}}(1+8t)^{-1/2} \cdot 8$

$= \frac{24}{\sqrt{1+8t}}$

set equal

$\frac{24}{\sqrt{1+8t}} = \frac{24}{5}$

$\sqrt{1+8t}^{\,2} = 5^2$

$\cancel{1} + 8t = 25$
$\cancel{-1} \qquad -1$

$\frac{\cancel{8}t}{\cancel{8}} = \frac{24}{8}$

$t = 3$

at $\boxed{3 \text{ sec}}$

3 is in the middle

b) slope of Q on $0 \le t \le 5$

(0,0) (5,20)

$\frac{20-0}{5-0} = \frac{20}{5} = \boxed{4 \text{ ft/sec}^2}$

c) Need position of both

position of P

$\int_0^{10} 6\left(\sqrt{1+8t} - 1\right) dt$

$\int_0^{10} 6\sqrt{1+8t} - 6$ $\qquad u = 1+8t$, $\frac{du}{8} = \frac{8}{8} dt$

$\frac{1}{8}\int_0^{10} 6\sqrt{u} - 6$

$\frac{1}{8}\left(\frac{6u^{3/2}}{3/2} - 6t\right)\Big|_0^{10}$

$\frac{1}{8}\left(4(1+8t)^{3/2} - 6t\right)\Big|_0^{10}$

$\frac{(1+8t)^{3/2}}{2} - 6t\Big|_0^{10}$

$\left(\frac{(81)^{3/2}}{2} - 60\right) - \left(\frac{1}{2} - 0\right)$

$\frac{729}{2} - \frac{60 \cdot 2}{1 \cdot 2} - \frac{1}{2} = \frac{608}{2} = \boxed{304}$

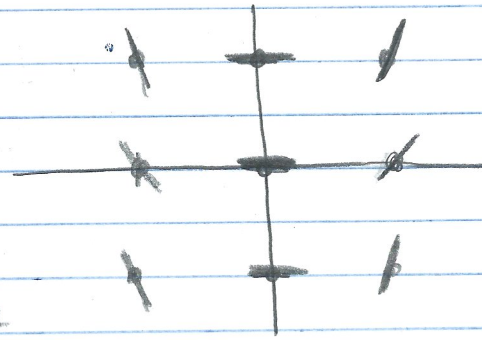
$\begin{array}{r} 729 \\ -121 \\ \hline 608 \end{array}$

$\boxed{\text{P wins}}$

position of Q - area under the curve

$\frac{1}{2}(5)(20) + \frac{1}{2}5(20+80) = 50 + 250 = 300$

(5) $\frac{dy}{dx} = 2x(y^2+1)$



a) $(0,0) = 2(0)(0^2+1) = 0$

$(1,0) = 2(1)(0^2+1) = 2(1) = 2$

$(1,1) = 2(1)(1^2+1) = 2(2) = 4$

$(1,-1) = 2(1)((-1)^2+1) = 2(2) = 4$

$(0,1) = 2(0)(1^2+1) = 0$

$(0,-1) = 2(0)((-1)^2+1) = 0$

$(-1,0) = 2(-1)(0^2+1)$
$= -2(1) = -2$

$(-1,1) = 2(-1)(1^2+1)$
$-2(2) = -4$

$(-1,-1) = 2(-1)((-1)^2+1)$
$-2(2) = -4$

b) $\frac{dy}{dx} = 2x(y^2+1)$

$\int \frac{dy}{y^2+1} = \int 2x\,dx$

$\tan^{-1}(y) = \frac{2x^2}{2} + C$

$\tan \tan^{-1}(y) = \tan(x^2+C)$

$y = \tan(x^2+c)$

plug in $(0,1)$

$1 = \tan(0^2+C)$

$\tan^{-1} 1 = \tan^{-1} \tan(C)$

$\tan^{-1}(1) = C$

$\pi/4 = C$

$y = \tan(x^2 + \frac{\pi}{4})$

vertical asymptotes at $x = \pm \pi/2$

$\cancel{-\frac{\pi}{2}} < x^2 + \frac{\pi}{4} < \pi/2$ $\frac{2\pi}{4}$

$\cancel{-\pi/4}$ $\quad -\pi/4 \quad -\pi/4$

$\sqrt{x^2} < \sqrt{\frac{\pi}{4}}$

Domain

$\boxed{|x| < \frac{\sqrt{\pi}}{2}}$

x-coor

6 a) $\int_0^2 f(t)\,dt = F(2) = \boxed{4}$

b) $\int_2^7 f(t)\,dt = F(7) - F(2) = 2 - 4 = \boxed{-2}$

c) $\frac{d}{dt} F(x) = \frac{d}{dt}\int_0^x f(t)\,dt$

$F'(x) = f(x) = 0$

so when does $F'(x) = 0$

slope $= 0$

$\boxed{x = 4}$

d) $F''(x) = f'(x) < 0$ when

when is it concave down

$\boxed{0 < t < 8}$

e) $G(x) = \int_2^x f(t)\,dt$

$= \int_2^0 f(t)\,dt + \int_0^x f(t)\,dt$

$= -\int_0^2 f(t)\,dt + \int_0^x f(x)$

$-F(2) \quad + F(x)$

$-4 \ + F(x)$

$F(x) - 4$ move down 4 units